

A Story of Functions®

Eureka Math™

Precalculus, Module 3

Student File_A

Contains copy-ready classwork and homework

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Printed in the U.S.A.

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10 9 8 7 6 5 4 3 2 1

Lesson 1: Solutions to Polynomial Equations

Classwork

Opening Exercise

How many solutions are there to the equation $x^2 = 1$? Explain how you know.

Example 1: Prove that a Quadratic Equation Has Only Two Solutions over the Set of Complex Numbers

Prove that 1 and -1 are the only solutions to the equation $x^2 = 1$.

Let $x = a + bi$ be a complex number so that $x^2 = 1$.

- Substitute $a + bi$ for x in the equation $x^2 = 1$.
- Rewrite both sides in standard form for a complex number.
- Equate the real parts on each side of the equation, and equate the imaginary parts on each side of the equation.
- Solve for a and b , and find the solutions for $x = a + bi$.

Exercises

Find the product.

1. $(z - 2)(z + 2)$

2. $(z + 3i)(z - 3i)$

Write each of the following quadratic expressions as the product of two linear factors.

3. $z^2 - 4$

4. $z^2 + 4$

5. $z^2 - 4i$

6. $z^2 + 4i$

7. Can a quadratic polynomial equation with real coefficients have one real solution and one complex solution? If so, give an example of such an equation. If not, explain why not.

Recall from Algebra II that every quadratic expression can be written as a product of two linear factors, that is,

$$ax^2 + bx + c = a(x - r_1)(x - r_2),$$

where r_1 and r_2 are solutions of the polynomial equation $ax^2 + bx + c = 0$.

8. Solve each equation by factoring, and state the solutions.

a. $x^2 + 25 = 0$

b. $x^2 + 10x + 25 = 0$

9. Give an example of a quadratic equation with $2 + 3i$ as one of its solutions.

10. A quadratic polynomial equation with real coefficients has a complex solution of the form $a + bi$ with $b \neq 0$. What must its other solution be and why?

11. Write the left side of each equation as a product of linear factors, and state the solutions.

a. $x^3 - 1 = 0$

b. $x^3 + 8 = 0$

c. $x^4 + 7x^2 + 10 = 0$

12. Consider the polynomial $p(x) = x^3 + 4x^2 + 6x - 36$.
- Graph $y = x^3 + 4x^2 + 6x - 36$, and find the real zero of polynomial p .
 - Write p as a product of linear factors.
 - What are the solutions to the equation $p(x) = 0$?
13. Malaya was told that the volume of a box that is a cube is 4,096 cubic inches. She knows the formula for the volume of a cube with side length x is $V(x) = x^3$, so she models the volume of the box with the equation $x^3 - 4096 = 0$.
- Solve this equation for x .
 - Malaya shows her work to Tiffany and tells her that she has found three different values for the side length of the box. Tiffany looks over Malaya's work and sees that it is correct but explains to her that there is only one valid answer. Help Tiffany explain which answer is valid and why.

14. Consider the polynomial $p(x) = x^6 - 2x^5 + 7x^4 - 10x^3 + 14x^2 - 8x + 8$.
- Graph $y = x^6 - 2x^5 + 7x^4 - 10x^3 + 14x^2 - 8x + 8$, and state the number of real zeros of p .
 - Verify that i is a zero of p .
 - Given that i is a zero of p , state another zero of p .
 - Given that $2i$ and $1 + i$ are also zeros of p , explain why polynomial p cannot possibly have any real zeros.
 - What is the solution set to the equation $p(x) = 0$?

15. Think of an example of a sixth-degree polynomial equation that, when written in standard form, has integer coefficients, four real number solutions, and two imaginary number solutions. How can you be sure your equation will have integer coefficients?

Lesson Summary

Relevant Vocabulary

POLYNOMIAL FUNCTION: Given a polynomial expression in one variable, a *polynomial function in one variable* is a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that for each real number x in the domain, $f(x)$ is the value found by substituting the number x into all instances of the variable symbol in the polynomial expression and evaluating.

It can be shown that if a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial function, then there is some nonnegative integer n and collection of real numbers $a_0, a_1, a_2, \dots, a_n$ with $a_n \neq 0$ such that the function satisfies the equation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

for every real number x in the domain, which is called the *standard form of the polynomial function*. The function $f(x) = 3x^3 + 4x^2 + 4x + 7$, where x can be any real number, is an example of a function written in standard form.

DEGREE OF A POLYNOMIAL FUNCTION: The *degree of a polynomial function* is the degree of the polynomial expression used to define the polynomial function. The degree is the highest degree of its terms.

The degree of $f(x) = 8x^3 + 4x^2 + 7x + 6$ is 3, but the degree of $g(x) = (x + 1)^2 - (x - 1)^2$ is 1 because when g is put into standard form, it is $g(x) = 4x$.

ZEROS OR ROOTS OF A FUNCTION: A *zero* (or *root*) of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is a number x of the domain such that $f(x) = 0$. A zero of a function is an element in the solution set of the equation $f(x) = 0$.

Given any two polynomial functions p and q , the solution set of the equation $p(x)q(x) = 0$ can be quickly found by solving the two equations $p(x) = 0$ and $q(x) = 0$ and combining the solutions into one set.

A number a is zero of a polynomial function p with multiplicity m if the factored form of p contains $(x - a)^m$.

Every polynomial function of degree n , for $n \geq 1$, has n zeros over the complex numbers, counted with multiplicity. Therefore, such polynomials can always be factored into n linear factors.

Problem Set

- Find all solutions to the following quadratic equations, and write each equation in factored form.

- $x^2 + 25 = 0$
- $-x^2 - 16 = -7$
- $(x + 2)^2 + 1 = 0$
- $(x + 2)^2 = x$
- $(x^2 + 1)^2 + 2(x^2 + 1) - 8 = 0$
- $(2x - 1)^2 = (x + 1)^2 - 3$
- $x^3 + x^2 - 2x = 0$
- $x^3 - 2x^2 + 4x - 8 = 0$

2. The following cubic equations all have at least one real solution. Find the remaining solutions.

a. $x^3 - 2x^2 - 5x + 6 = 0$

b. $x^3 - 4x^2 + 6x - 4 = 0$

c. $x^3 + x^2 + 9x + 9 = 0$

d. $x^3 + 4x = 0$

e. $x^3 + x^2 + 2x + 2 = 0$

3. Find the solutions of the following equations.

a. $4x^4 - x^2 - 18 = 0$

b. $x^3 - 8 = 0$

c. $8x^3 - 27 = 0$

d. $x^4 - 1 = 0$

e. $81x^4 - 64 = 0$

f. $20x^4 + 121x^2 - 25 = 0$

g. $64x^3 + 27 = 0$

h. $x^3 + 125 = 0$

Lesson 2: Does Every Complex Number Have a Square Root?

Classwork

Exercises 1–6

1. Use the geometric effect of complex multiplication to describe how to calculate a square root of $z = 119 + 120i$.
2. Calculate an estimate of a square root of $119 + 120i$.
3. Every real number has two square roots. Explain why.
4. Provide a convincing argument that every complex number must also have two square roots.
5. Explain how the polynomial identity $x^2 - b = (x - \sqrt{b})(x + \sqrt{b})$ relates to the argument that every number has two square roots.

Exercises 7–9

7. Use the method in the Example to find the square roots of $1 + \sqrt{3}i$.

8. Find the square roots of each complex number.

a. $5 + 12i$

b. $5 - 12i$

9. Show that if $p + qi$ is a square root of $z = a + bi$, then $p - qi$ is a square root of the conjugate of z , $\bar{z} = a - bi$.
- a. Explain why $(p + qi)^2 = a + bi$.

b. What do a and b equal in terms of p and q ?

c. Calculate $(p - qi)^2$. What is the real part, and what is the imaginary part?

d. Explain why $(p - qi)^2 = a - bi$.

Lesson Summary

The square roots of a complex number $a + bi$ are of the form $p + qi$ and $-p - qi$ and can be found by solving the equations $p^2 - q^2 = a$ and $2pq = b$.

Problem Set

Find the two square roots of each complex number by creating and solving polynomial equations.

1. $z = 15 - 8i$
2. $z = 8 - 6i$
3. $z = -3 + 4i$
4. $z = -5 - 12i$
5. $z = 21 - 20i$
6. $z = 16 - 30i$
7. $z = i$

A *Pythagorean triple* is a set of three positive integers a , b , and c such that $a^2 + b^2 = c^2$. Thus, these integers can be the lengths of the sides of a right triangle.

8. Show algebraically that for positive integers p and q , if

$$a = p^2 - q^2$$

$$b = 2pq$$

$$c = p^2 + q^2$$

then $a^2 + b^2 = c^2$.

9. Select two integers p and q , use the formulas in Problem 8 to find a , b , and c , and then show those numbers satisfy the equation $a^2 + b^2 = c^2$.
10. Use the formulas from Problem 8, and find values for p and q that give the following famous triples.
- a. (3, 4, 5)
 - b. (5, 12, 13)
 - c. (7, 24, 25)
 - d. (9, 40, 41)
11. Is it possible to write the Pythagorean triple (6, 8, 10) in the form $a = p^2 - q^2$, $b = 2pq$, $c = p^2 + q^2$ for some integers p and q ? Verify your answer.
12. Choose your favorite Pythagorean triple (a, b, c) that has a and b sharing only 1 as a common factor, for example (3, 4, 5), (5, 12, 13), or (7, 24, 25), ... Find the square of the length of a square root of $a + bi$; that is, find $|p + qi|^2$, where $p + qi$ is a square root of $a + bi$. What do you observe?

Lesson 3: Roots of Unity

Classwork

Opening Exercise

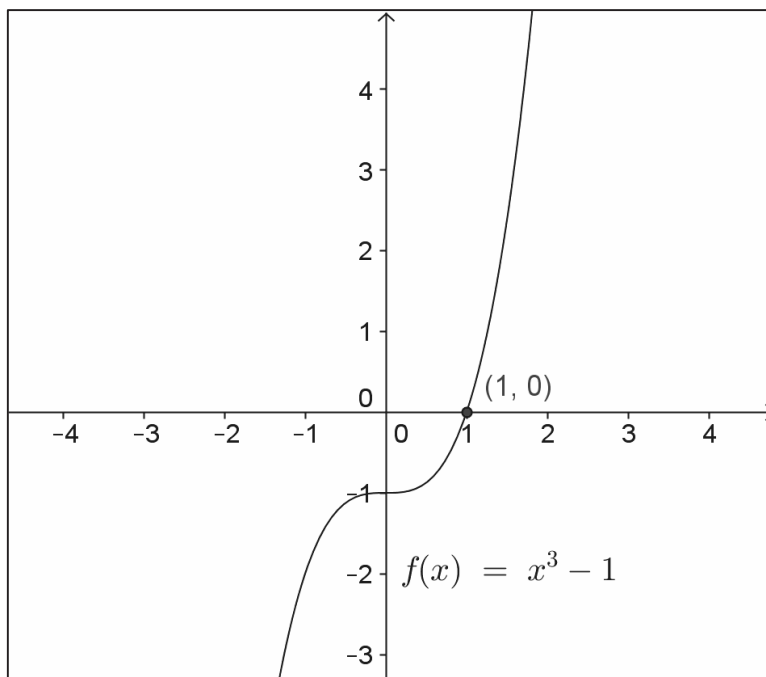
Consider the equation $x^n = 1$ for positive integers n .

- a. Must an equation of this form have a real solution? Explain your reasoning.
- b. Could an equation of this form have two real solutions? Explain your reasoning.
- c. How many complex solutions are there for an equation of this form? Explain how you know.

Exploratory Challenge

Consider the equation $x^3 = 1$.

- a. Use the graph of $f(x) = x^3 - 1$ to explain why 1 is the only real number solution to the equation $x^3 = 1$.



- b. Find all of the complex solutions to the equation $x^3 = 1$. Come up with as many methods as you can for finding the solutions to this equation.

Exercises

Solutions to the equation $x^n = 1$ for positive integers n are called *the n^{th} roots of unity*.

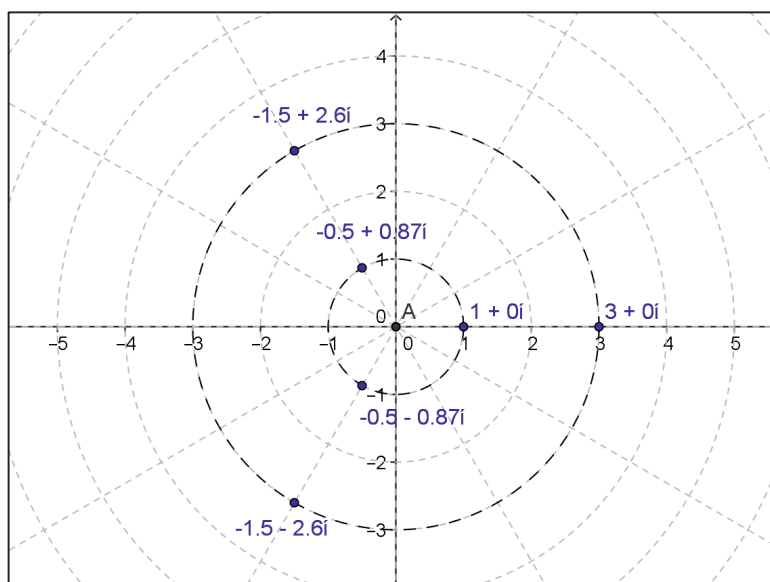
1. What are the square roots of unity in rectangular and polar form?
2. What are the fourth roots of unity in rectangular and polar form? Solve this problem by creating and solving a polynomial equation. Show work to support your answer.
3. Find the sixth roots of unity in rectangular form by creating and solving a polynomial equation. Show work to support your answer. Find the sixth roots of unity in polar form.
4. Without using a formula, what would be the polar forms of the fifth roots of unity? Explain using the geometric effect of multiplication complex numbers.

Discussion

What is the modulus of each root of unity regardless of the value of n ? Explain how you know.

How could you describe the location of the roots of unity in the complex plane?

The diagram below shows the solutions to the equation $x^3 = 27$. How do these numbers compare to the cube roots of unity (e.g., the solutions to $x^3 = 1$)?



Lesson Summary

The solutions to the equation $x^n = 1$ for positive integers n are called the n^{th} roots of unity. For any value of $n > 2$, the roots of unity are complex numbers of the form $z_k = a_k + b_k i$ for positive integers $1 < k < n$ with the corresponding points (a_k, b_k) at the vertices of a regular n -gon centered at the origin with one vertex at $(1,0)$.

The fundamental theorem of algebra guarantees that an equation of the form $x^n = k$ will have n complex solutions. If n is odd, then the real number $\sqrt[n]{k}$ is the only real solution. If n is even, then the equation has exactly two real solutions: $\sqrt[n]{k}$ and $-\sqrt[n]{k}$.

Given a complex number z with modulus r and argument θ , the n^{th} roots of z are given by

$$\sqrt[n]{r} \left(\cos \left(\frac{\theta}{n} + \frac{2\pi k}{n} \right) + i \sin \left(\frac{\theta}{n} + \frac{2\pi k}{n} \right) \right)$$

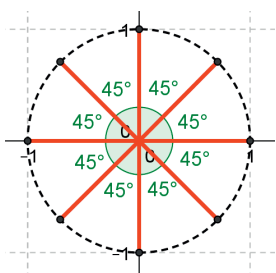
for integers k and n such that $n > 0$ and $0 \leq k < n$.

Problem Set

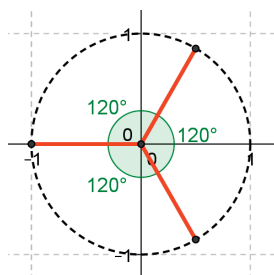
- Graph the n^{th} roots of unity in the complex plane for the specified value of n .
 - $n = 3$
 - $n = 4$
 - $n = 5$
 - $n = 6$
- Find the cube roots of unity by using each method stated.
 - Solve the polynomial equation $x^3 = 1$ algebraically.
 - Use the polar form $z^3 = r(\cos(\theta) + i\sin(\theta))$, and find the modulus and argument of z .
 - Solve $(a + bi)^3 = 1$ by expanding $(a + bi)^3$ and setting it equal to $1 + 0i$.
- Find the fourth roots of unity by using the method stated.
 - Solve the polynomial equation $x^4 = 1$ algebraically.
 - Use the polar form $z^4 = r(\cos(\theta) + i\sin(\theta))$, and find the modulus and argument of z .
 - Solve $(a + bi)^4 = 1$ by expanding $(a + bi)^4$ and setting it equal to $1 + 0i$.
- Find the fifth roots of unity by using the method stated.

Use the polar form $z^5 = r(\cos(\theta) + i\sin(\theta))$, and find the modulus and argument of z .

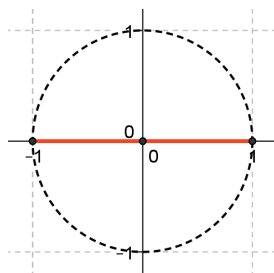
5. Find the sixth roots of unity by using the method stated.
- Solve the polynomial equation $x^6 = 1$ algebraically.
 - Use the polar form $z^6 = r(\cos(\theta) + \sin(\theta))$, and find the modulus and argument of z .
6. Consider the equation $x^N = 1$ where N is a positive whole number.
- For which value of N does $x^N = 1$ have only one solution?
 - For which value of N does $x^N = 1$ have only ± 1 as solutions?
 - For which value of N does $x^N = 1$ have only ± 1 and $\pm i$ as solutions?
 - For which values of N does $x^N = 1$ have ± 1 as solutions?
7. Find the equation that has the following solutions.
-



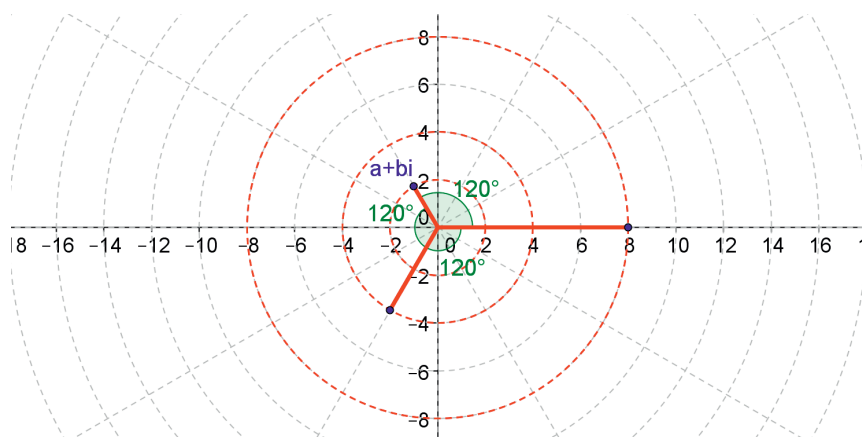
b.



c.



8. Find the equation $(a + bi)^N = c$ that has solutions shown in the graph below.



Lesson 4: The Binomial Theorem

Classwork

Exercises

1. Show that $z = 1 + i$ is a solution to the fourth degree polynomial equation $z^4 - z^3 + 3z^2 - 4z + 6 = 0$.
2. Show that $z = 1 - i$ is a solution to the fourth degree polynomial equation $z^4 - z^3 + 3z^2 - 4z + 6 = 0$.
3. Based on the patterns seen in Pascal's triangle, what would be the coefficients of Rows 7 and 8 in the triangle? Write the coefficients of the triangle beneath the part of the triangle shown.

Row 0:						1											
Row 1:						1		1									
Row 2:						1		2		1							
Row 3:						1		3		3		1					
Row 4:						1		4		6		4		1			
Row 5:						1		5		10		10		5		1	
Row 6:						1		6		15		20		15		6	
Row 7:																	
Row 8:																	

4. Calculate the following factorials.

a. $6!$

b. $10!$

5. Calculate the value of the following factorial expressions.

a. $\frac{7!}{6!}$

b. $\frac{10!}{6!}$

c. $\frac{8!}{5!}$

d. $\frac{12!}{10!}$

6. Calculate the following quantities.

a. $C(1,0)$ and $C(1,1)$

b. $C(2,0)$, $C(2,1)$, and $C(2,2)$

c. $C(3,0)$, $C(3,1)$, $C(3,2)$, and $C(3,3)$

d. $C(4,0)$, $C(4,1)$, $C(4,2)$, $C(4,3)$, and $C(4,4)$

7. What patterns do you see in Exercise 6?

8. Expand the expression $(u + v)^3$.
9. Expand the expression $(u + v)^4$.
- 10.
- Multiply the expression you wrote in Exercise 9 by u .
 - Multiply the expression you wrote in Exercise 9 by v .
 - How can you use the results from parts (a) and (b) to find the expanded form of the expression $(u + v)^5$?
11. What do you notice about your expansions for $(u + v)^4$ and $(u + v)^5$? Does your observation hold for other powers of $(u + v)$?

12. Use the binomial theorem to expand the following binomial expressions.

a. $(x + y)^6$

b. $(x + 2y)^3$

c. $(ab + bc)^4$

d. $(3xy - 2z)^3$

e. $(4p^2qr - qr^2)^5$

Lesson Summary

Pascal's triangle is an arrangement of numbers generated recursively:

Row 0:				1			
Row 1:			1		1		
Row 2:			1		2		1
Row 3:			1		3		3
Row 4:			1		4		6
Row 5:			1		5		10
			1		6		15
			1		10		21
			1		15		28
			1		21		36
			1		28		45
			1		36		56
			1		45		63
			1		56		70
			1		63		77
			1		70		84
			1		77		91
			1		84		99
			1		91		108
			1		99		117
			1		108		126
			1		117		136
			1		126		147
			1		136		158
			1		147		170
			1		158		183
			1		170		196
			1		183		210
			1		196		225
			1		210		241
			1		225		258
			1		241		276
			1		258		295
			1		276		315
			1		295		336
			1		315		358
			1		336		381
			1		358		405
			1		381		430
			1		405		456
			1		430		483
			1		456		511
			1		483		540
			1		511		570
			1		540		601
			1		570		633
			1		601		666
			1		633		700
			1		666		735
			1		700		771
			1		735		808
			1		771		846
			1		808		885
			1		846		925
			1		885		966
			1		925		1008
			1		966		1051
			1		1008		1105
			1		1051		1160
			1		1105		1216
			1		1160		1273
			1		1216		1331
			1		1273		1390
			1		1331		1450
			1		1390		1511
			1		1450		1573
			1		1511		1636
			1		1573		1700
			1		1636		1765
			1		1700		1831
			1		1765		1900
			1		1831		1970
			1		1900		2041
			1		1970		2113
			1		2041		2186
			1		2113		2260
			1		2186		2335
			1		2260		2411
			1		2335		2488
			1		2411		2566
			1		2488		2645
			1		2566		2725
			1		2645		2806
			1		2725		2888
			1		2806		2971
			1		2888		3055
			1		2971		3140
			1		3055		3226
			1		3140		3313
			1		3226		3401
			1		3313		3490
			1		3401		3580
			1		3490		3671
			1		3580		3763
			1		3671		3856
			1		3763		3950
			1		3856		4045
			1		3950		4141
			1		4045		4238
			1		4141		4336
			1		4238		4435
			1		4336		4535
			1		4435		4636
			1		4535		4738
			1		4636		4841
			1		4738		4945
			1		4841		5050
			1		4945		5156
			1		5050		5263
			1		5156		5371
			1		5263		5480
			1		5371		5590
			1		5480		5701
			1		5590		5813
			1		5701		5926
			1		5813		6040
			1		5926		6155
			1		6040		6271
			1		6155		6388
			1		6271		6506
			1		6388		6625
			1		6506		6745
			1		6625		6866
			1		6745		6988
			1		6866		7111
			1		6988		7235
			1		7111		7360
			1		7235		7486
			1		7360		7613
			1		7486		7741
			1		7613		7870
			1		7741		8000
			1		7870		8131
			1		8000		8263
			1		8131		8396
			1		8263		8530
			1		8396		8665
			1		8530		8801
			1		8665		8938
			1		8801		9076
			1		8938		9215
			1		9076		9355
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			1		9638		9925
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			1		10511		10810
			1		10660		10961
			1		10810		11113
			1		10961		11266
			1		11113		11420
			1		11266		11575
			1		11420		11731
			1		11575		11888
			1		11731		12046
			1		11888		12205
			1		12046		12365
			1		12205		12526
			1		12365		12688
			1		12526		12851
			1		12688		13015
			1		12851		13180
			1		13015		13346
			1		13180		13513
			1		13346		13681
			1		13513		13850
			1		13681		14020
			1		13850		14191
			1		14020		14363
			1		14191		14536
			1		14363		14710
			1		14536		14885
			1		14710		15061
			1		14885		15238
			1		15061		15416
			1		15238		15595
			1		15416		15775
			1		15595		15956
			1		15775		16138
			1		15956		16321
			1		16138		16505
			1		16321		16690
			1		16505		16876
			1		16690		17063
			1		16876		17251
			1		17063		17440
			1		17251		17630
			1		17440		17821
			1		17630		18013
			1		17821		18206
			1		18013		18400
			1		18206		18595
			1		18400		18791
			1		18595		18988
			1		18791		19186
			1		18988		19385
			1		19186		19585
			1		19385		19786
			1		19585		19988
			1		19786		20191
			1		19988		20395
			1		20191		20600
			1		20395		20806
			1		20600		21013
			1		20806		21221
			1		21013		21430
			1		21221		21640
			1		21430		21851
			1		21640		22063
			1		21851		22276
			1		22063		22490
			1		22276		22705
			1		22490		22921
			1		22705		23138
			1		22921		23356
			1		23138		23575
			1		23356		23795
			1		23575		24016
			1		23795		24238
			1		24016		24461
			1		24238		24685
			1		24461		24910
			1		24685		25136
			1		24910		25363
			1		25136		25591
			1		25363		25820
			1		25591		26050
			1		25820		26281
			1		26050		26513
			1		26281		26746
			1		26513		26980
			1		26746		27215
			1		26980		27451
			1		27215		27688
			1		27451		27926
			1		27688		28165
			1		27926</		

3. Use the binomial theorem to expand the following binomial expressions.
- $(1 + \sqrt{2})^5$
 - $(1 + i)^9$
 - $(1 - \pi)^5$ (Hint: $1 - \pi = 1 + (-\pi)$.)
 - $(\sqrt{2} + i)^6$
 - $(2 - i)^6$
4. Consider the expansion of $(a + b)^{12}$. Determine the coefficients for the terms with the powers of a and b shown.
- a^2b^{10}
 - a^5b^7
 - a^8b^4
5. Consider the expansion of $(x + 2y)^{10}$. Determine the coefficients for the terms with the powers of x and y shown.
- x^2y^8
 - x^4y^6
 - x^5y^5
6. Consider the expansion of $(5p + 2q)^6$. Determine the coefficients for the terms with the powers of p and q shown.
- p^2q^4
 - p^5q
 - p^3q^3
7. Explain why the coefficient of the term that contains u^n is 1 in the expansion of $(u + v)^n$.
8. Explain why the coefficient of the term that contains $u^{n-1}v$ is n in the expansion of $(u + v)^n$.
9. Explain why the rows of Pascal's triangle are symmetric. That is, explain why $C(n, k) = C(n, (n - k))$.

Lesson 5: The Binomial Theorem

Classwork

Opening Exercise

Write the first six rows of Pascal's triangle. Then, use the triangle to find the coefficients of the terms with the powers of u and v shown, assuming that all expansions are in the form $(u + v)^n$. Explain how Pascal's triangle allows you to determine the coefficient.

a. u^2v^4

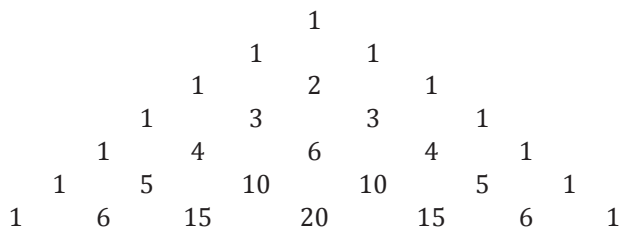
b. u^3v^2

c. u^2v^2

d. v^{10}

Example 1

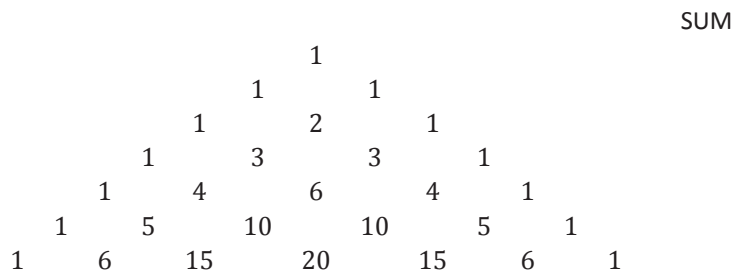
Look at the alternating sums of the rows of Pascal's triangle. An alternating sum alternately subtracts and then adds values. For example, the alternating sum of Row 2 would be $1 - 2 + 1$, and the alternating sum of Row 3 would be $1 - 3 + 3 - 1$.



- Compute the alternating sum for each row of the triangle shown.
- Use the binomial theorem to explain why each alternating sum of a row in Pascal's triangle is 0.

Exercises

1. Consider Rows 0–6 of Pascal’s triangle.
 - a. Find the sum of each row.



- b. What pattern do you notice in the sums computed?
 - c. Use the binomial theorem to explain this pattern.
2. Consider the expression 11^n .
 - a. Calculate 11^n , where $n = 0, 1, 2, 3, 4$.
 - b. What pattern do you notice in the successive powers?

- c. Use the binomial theorem to demonstrate why this pattern arises.
- d. Use a calculator to find the value of 11^5 . Explain whether this value represents what would be expected based on the pattern seen in lower powers of 11.

Example 2

We know that the volume $V(r)$ and surface area $S(r)$ of a sphere of radius r are given by these formulas:

$$V(r) = \frac{4}{3}\pi r^3$$

$$S(r) = 4\pi r^2$$

Suppose we increase the radius of a sphere by 0.01 units from r to $r + 0.01$.

- a. Use the binomial theorem to write an expression for the increase in volume.
- b. Write an expression for the average rate of change of the volume as the radius increases from r to $r + 0.01$.

- c. Simplify the expression in part (b) to compute the average rate of change of the volume of a sphere as the radius increases from r to $r + 0.01$.
- d. What does the expression from part (c) resemble?
- e. Why does it make sense that the average rate of change should approximate the surface area? Think about the geometric figure formed by $V(r + 0.01) - V(r)$. What does this represent?
- f. How could we approximate the volume of the shell using surface area? And the average rate of change for the volume?

Problem Set

- Consider the binomial $(2u - 3v)^6$.
 - Find the term that contains v^4 .
 - Find the term that contains u^3 .
 - Find the third term.
- Consider the binomial $(u^2 - v^3)^6$.
 - Find the term that contains v^6 .
 - Find the term that contains u^6 .
 - Find the fifth term.
- Find the sum of all coefficients in the following binomial expansion.
 - $(2u + v)^{10}$
 - $(2u - v)^{10}$
 - $(2u - 3v)^{11}$
 - $(u - 3v)^{11}$
 - $(1 + i)^{10}$
 - $(1 - i)^{10}$
 - $(1 + i)^{200}$
 - $(1 + i)^{201}$
- Expand the binomial $(1 + \sqrt{2}i)^6$.
- Show that $(2 + \sqrt{2}i)^{20} + (2 - \sqrt{2}i)^{20}$ is an integer.
- We know $(u + v)^2 = u^2 + 2uv + v^2 = u^2 + v^2 + 2uv$. Use this pattern to predict what the expanded form of each expression would be. Then, expand the expression, and compare your results.
 - $(u + v + w)^2$
 - $(a + b + c + d)^2$
- Look at the powers of 101 up to the fourth power on a calculator. Explain what you see. Predict the value of 101^5 , and then find the answer on a calculator. Are they the same?
- Can Pascal's triangle be applied to $\left(\frac{1}{u} + \frac{1}{v}\right)^n$ given $u, v \neq 0$?

9. The volume and surface area of a sphere are given by $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. Suppose we increase the radius of a sphere by 0.001 units from r to $r + 0.001$.
- Use the binomial theorem to write an expression for the increase in volume $V(r + 0.001) - V(r)$ as the sum of three terms.
 - Write an expression for the average rate of change of the volume as the radius increases from r to $r + 0.001$.
 - Simplify the expression in part (b) to compute the average rate of change of the volume of a sphere as the radius increases from r to $r + 0.001$.
 - What does the expression from part (c) resemble?
 - Why does it make sense that the average rate of change should approximate the surface area? Think about the geometric figure formed by $V(r + 0.001) - V(r)$. What does this represent?
 - How could we approximate the volume of the shell using surface area? And the average rate of change for the volume?
 - Find the difference between the average rate of change of the volume and $S(r)$ when $r = 1$.
10. The area and circumference of a circle of radius r are given by $A(r) = \pi r^2$ and $C(r) = 2\pi r$. Suppose we increase the radius of a sphere by 0.001 units from r to $r + 0.001$.
- Use the binomial theorem to write an expression for the increase in area volume $A(r + 0.001) - A(r)$ as a sum of three terms.
 - Write an expression for the average rate of change of the area as the radius increases from r to $r + 0.001$.
 - Simplify the expression in part (b) to compute the average rate of change of the area of a circle as the radius increases from r to $r + 0.001$.
 - What does the expression from part (c) resemble?
 - Why does it make sense that the average rate of change should approximate the area of a circle? Think about the geometric figure formed by $A(r + 0.001) - A(r)$. What does this represent?
 - How could we approximate the area of the shell using circumference? And the average rate of change for the area?
 - Find the difference between the average rate of change of the area and $C(r)$ when $r = 1$.

Lesson 6: Curves in the Complex Plane

Classwork

Opening Exercise

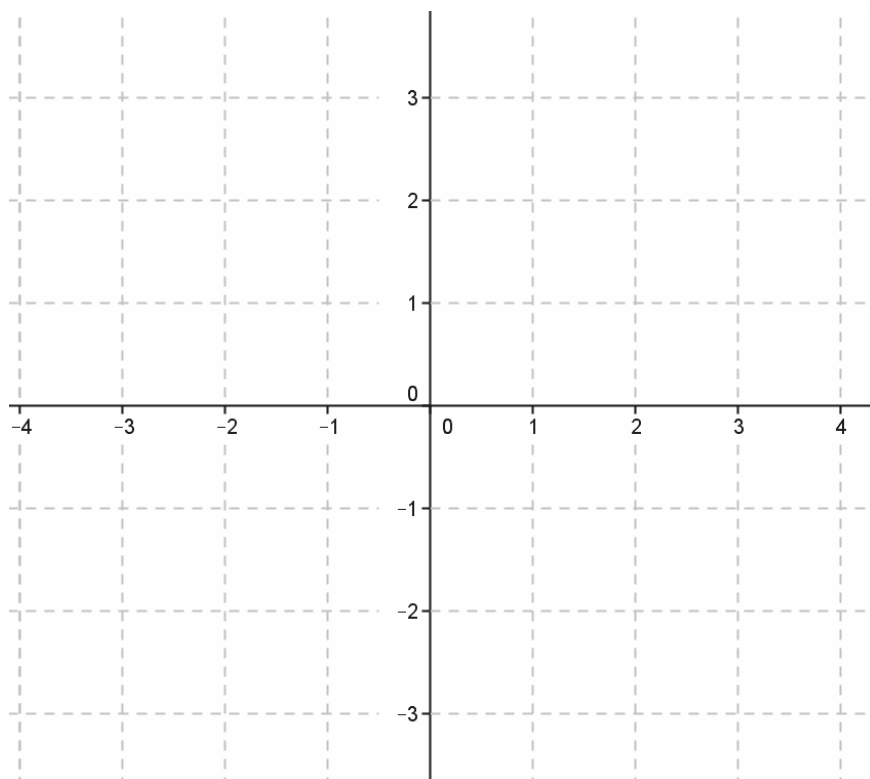
- a. Consider the complex number $z = a + bi$.
 - i. Write z in polar form. What do the variables represent?
 - ii. If $r = 3$ and $\theta = 90^\circ$, where would z be plotted in the complex plane?
 - iii. Use the conditions in part (ii) to write z in rectangular form. Explain how this representation corresponds to the location of z that you found in part (ii).
- b. Recall the set of points defined by $z = 3(\cos(\theta) + i \sin(\theta))$ for $0^\circ \leq \theta < 360^\circ$, where θ is measured in degrees.
 - i. What does z represent graphically? Why?
 - ii. What does z represent geometrically?

- c. Consider the set of points defined by $z = 5 \cos(\theta) + 3i \sin(\theta)$.
- i. Plot z for $\theta = 0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$. Based on your plot, form a conjecture about the graph of the set of complex numbers.
- ii. Compare this graph to the graph of $z = 3(\cos(\theta) + i \sin(\theta))$. Form a conjecture about what accounts for the differences between the graphs.

Example 1

Consider again the set of complex numbers represented by $z = 3(\cos(\theta) + i \sin(\theta))$ for $0^\circ \leq \theta < 360^\circ$.

θ	$3 \cos(\theta)$	$3 \sin(\theta)$	$(3 \cos(\theta), 3 \sin(\theta))$
0			
$\frac{\pi}{4}$			
$\frac{\pi}{2}$			
$\frac{3\pi}{4}$			
π			
$\frac{5\pi}{4}$			
$\frac{3\pi}{2}$			
$\frac{7\pi}{4}$			
2π			



- a. Use an ordered pair to write a representation for the points defined by z as they would be represented in the coordinate plane.
- b. Write an equation that is true for all the points represented by the ordered pair you wrote in part (a).
- c. What does the graph of this equation look like in the coordinate plane?

Exercises 1–2

1. Recall the set of points defined by $z = 5 \cos(\theta) + 3i \sin(\theta)$.
 - a. Use an ordered pair to write a representation for the points defined by z as they would be represented in the coordinate plane.
 - b. Write an equation in the coordinate plane that is true for all the points represented by the ordered pair you wrote in part (a).

2. Find an algebraic equation for all the points in the coordinate plane traced by the complex numbers $z = \sqrt{2} \cos(\theta) + i \sin(\theta)$.

Example 2

The equation of an ellipse is given by $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

- a. Sketch the graph of the ellipse.

Exercise 3

3. The equation of an ellipse is given by $\frac{x^2}{9} + \frac{y^2}{26} = 1$.

a. Sketch the graph of the ellipse.

b. Rewrite the equation of the ellipse in complex form.

Example 3

A set of points in the complex plane can be represented in the complex plane as $z = 2 + i + 7 \cos(\theta) + i \sin(\theta)$ as θ varies.

a. Find an algebraic equation for the points described.

b. Sketch the graph of the ellipse.

Problem Set

1. Write the real form of each complex equation.

- $z = 4 \cos(\theta) + 9i \sin(\theta)$
- $z = 6 \cos(\theta) + i \sin(\theta)$
- $z = \sqrt{5} \cos(\theta) + \sqrt{10}i \sin(\theta)$
- $z = 5 - 2i + 4 \cos(\theta) + 7i \sin(\theta)$

2. Sketch the graphs of each equation.

- $z = 3 \cos(\theta) + i \sin(\theta)$
- $z = -2 + 3i + 4 \cos(\theta) + i \sin(\theta)$
- $\frac{(x-1)^2}{9} + \frac{y^2}{25} = 1$
- $\frac{(x-2)^2}{3} + \frac{y^2}{15} = 1$

3. Write the complex form of each equation.

- $\frac{x^2}{16} + \frac{y^2}{36} = 1$
- $\frac{x^2}{400} + \frac{y^2}{169} = 1$
- $\frac{x^2}{19} + \frac{y^2}{2} = 1$
- $\frac{(x-3)^2}{100} + \frac{(y+5)^2}{16} = 1$

4. Carrie converted the equation $z = 7 \cos(\theta) + 4i \sin(\theta)$ to the real form $\frac{x^2}{7} + \frac{y^2}{4} = 1$. Her partner Ginger said that the ellipse must pass through the point $(7 \cos(0), 4 \sin(0)) = (7, 0)$ and this point does not satisfy Carrie's equation, so the equation must be wrong. Who made the mistake, and what was the error? Explain how you know.
5. Cody says that the center of the ellipse with complex equation $z = 4 - 5i + 2 \cos(\theta) + 3i \sin(\theta)$ is $(4, -5)$, while his partner, Jarrett, says that the center of this ellipse is $(-4, 5)$. Which student is correct? Explain how you know.

Extension:

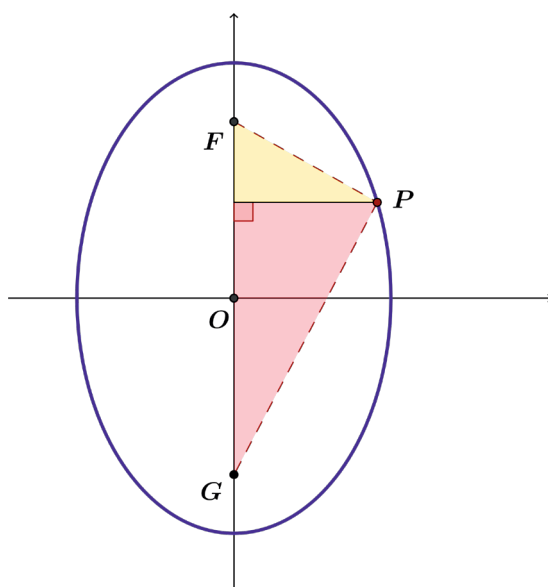
6. Any equation of the form $ax^2 + bx + cy^2 + dy + e = 0$ with $a > 0$ and $c > 0$ might represent an ellipse. The equation $4x^2 + 8x + 3y^2 + 12y + 4 = 0$ is such an equation of an ellipse.
- Rewrite the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ in standard form to locate the center of the ellipse (h, k) .
 - Describe the graph of the ellipse, and then sketch the graph.
 - Write the complex form of the equation for this ellipse.

Lesson 7: Curves from Geometry

Classwork

Exercise

Points F and G are located at $(0, 3)$ and $(0, -3)$. Let $P(x, y)$ be a point such that $PF + PG = 8$. Use this information to show that the equation of the ellipse is $\frac{x^2}{7} + \frac{y^2}{16} = 1$.



Problem Set

- Derive the equation of the ellipse with the given foci F and G that passes through point P . Write your answer in standard form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
 - The foci are $F(-2,0)$ and $G(2,0)$, and point $P(x,y)$ satisfies the condition $PF + PG = 5$.
 - The foci are $F(-1,0)$ and $G(1,0)$, and point $P(x,y)$ satisfies the condition $PF + PG = 5$.
 - The foci are $F(0,-1)$ and $G(0,1)$, and point $P(x,y)$ satisfies the condition $PF + PG = 4$.
 - The foci are $F(-\frac{2}{3}, 0)$ and $G(\frac{2}{3}, 0)$, and point $P(x,y)$ satisfies the condition $PF + PG = 3$.
 - The foci are $F(0,-5)$ and $G(0,5)$, and point $P(x,y)$ satisfies the condition $PF + PG = 12$.
 - The foci are $F(-6,0)$ and $G(6,0)$, and point $P(x,y)$ satisfies the condition $PF + PG = 20$.
- Recall from Lesson 6 that the semi-major axes of an ellipse are the segments from the center to the farthest vertices, and the semi-minor axes are the segments from the center to the closest vertices. For each of the ellipses in Problem 1, find the lengths a and b of the semi-major axes.
- Summarize what you know about equations of ellipses centered at the origin with vertices $(a, 0)$, $(-a, 0)$, $(0, b)$, and $(0, -b)$.
- Use your answer to Problem 3 to find the equation of the ellipse for each of the situations below.
 - An ellipse centered at the origin with x -intercepts $(-2,0)$, $(2,0)$ and y -intercepts $(0,8)$, $(0,-8)$
 - An ellipse centered at the origin with x -intercepts $(-\sqrt{5}, 0)$, $(\sqrt{5}, 0)$ and y -intercepts $(0,3)$, $(0,-3)$
- Examine the ellipses and the equations of the ellipses you have worked with, and describe the ellipses with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the three cases $a > b$, $a = b$, and $b > a$.
- Is it possible for $\frac{x^2}{4} + \frac{y^2}{9} = 1$ to have foci at $(-c, 0)$ and $(c, 0)$ for some real number c ?
- For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation $\frac{x^2}{4} + y^2 = k$.
 - $k = 1$
 - $k = \frac{1}{4}$
 - $k = \frac{1}{9}$
 - $k = \frac{1}{16}$
 - $k = \frac{1}{25}$

- f. $k = \frac{1}{100}$
- g. Make a conjecture: Which points in the plane satisfy the equation $\frac{x^2}{4} + y^2 = 0$?
- h. Explain why your conjecture in part (g) makes sense algebraically.
- i. Which points in the plane satisfy the equation $\frac{x^2}{4} + y^2 = -1$?
8. For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation $\frac{x^2}{k} + y^2 = 1$.
- a. $k = 1$
- b. $k = 2$
- c. $k = 4$
- d. $k = 10$
- e. $k = 25$
- f. Describe what happens to the graph of $\frac{x^2}{k} + y^2 = 1$ as $k \rightarrow \infty$.
9. For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation $x^2 + \frac{y^2}{k} = 1$.
- a. $k = 1$
- b. $k = 2$
- c. $k = 4$
- d. $k = 10$
- e. $k = 25$
- f. Describe what happens to the graph of $x^2 + \frac{y^2}{k} = 1$ as $k \rightarrow \infty$.

Lesson 8: Curves from Geometry

Classwork

Exercises

1. Let $F(0,5)$ and $G(0,-5)$ be the foci of a hyperbola. Let the points $P(x,y)$ on the hyperbola satisfy either $PF - PG = 6$ or $PG - PF = 6$. Use the distance formula to derive an equation for this hyperbola, writing your answer in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
2. Where does the hyperbola described above intersect the y -axis?
3. Find an equation for the line that acts as a boundary for the portion of the curve that lies in the first quadrant.
4. Sketch the graph of the hyperbola described above.

Problem Set

1. For each hyperbola described below: (1) Derive an equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. (2) State any x - or y -intercepts. (3) Find the equations for the asymptotes of the hyperbola.
 - a. Let the foci be $A(-2,0)$ and $B(2,0)$, and let P be a point for which either $PA - PB = 2$ or $PB - PA = 2$.
 - b. Let the foci be $A(-5,0)$ and $B(5,0)$, and let P be a point for which either $PA - PB = 5$ or $PB - PA = 5$.
 - c. Consider $A(0, -3)$ and $B(0,3)$, and let P be a point for which either $PA - PB = 2.5$ or $PB - PA = 2.5$.
 - d. Consider $A(0, -\sqrt{2})$ and $B(0, \sqrt{2})$, and let P be a point for which either $PA - PB = 2$ or $PB - PA = 2$.
2. Graph the hyperbolas in parts (a)–(d) in Problem 1.
3. For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation $x^2 - y^2 = k$.
 - a. $k = 4$
 - b. $k = 1$
 - c. $k = \frac{1}{4}$
 - d. $k = 0$
 - e. $k = -\frac{1}{4}$
 - f. $k = -1$
 - g. $k = -4$
 - h. Describe the hyperbolas $x^2 - y^2 = k$ for different values of k . Consider both positive and negative values of k , and consider values of k close to zero and far from zero.
 - i. Are there any values of k so that the equation $x^2 - y^2 = k$ has no solution?
4. For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation $\frac{x^2}{k} - y^2 = 1$.
 - a. $k = -1$
 - b. $k = 1$
 - c. $k = 2$
 - d. $k = 4$
 - e. $k = 10$
 - f. $k = 25$
 - g. Describe what happens to the graph of $\frac{x^2}{k} - y^2 = 1$ as $k \rightarrow \infty$.

5. For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation

$$x^2 - \frac{y^2}{k} = 1.$$

- a. $k = -1$
- b. $k = 1$
- c. $k = 2$
- d. $k = 4$
- e. $k = 10$

- f. Describe what happens to the graph $x^2 - \frac{y^2}{k} = 1$ as $k \rightarrow \infty$.

6. An equation of the form $ax^2 + bx + cy^2 + dy + e = 0$ where a and c have opposite signs might represent a hyperbola.

- a. Apply the process of completing the square in both x and y to convert the equation

$$9x^2 - 36x - 4y^2 - 8y - 4 = 0 \text{ to one of the standard forms for a hyperbola: } \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ or } \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1.$$

- b. Find the center of this hyperbola.
- c. Find the asymptotes of this hyperbola.
- d. Graph the hyperbola.

7. For each equation below, identify the graph as either an ellipse, a hyperbola, two lines, or a single point. If possible, write the equation in the standard form for either an ellipse or a hyperbola.

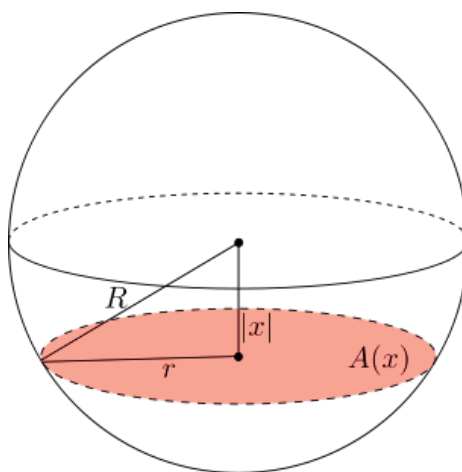
- a. $4x^2 - 8x + 25y^2 - 100y + 4 = 0$
- b. $4x^2 - 16x - 9y^2 - 54y - 65 = 0$
- c. $4x^2 + 8x + y^2 + 2y + 5 = 0$
- d. $-49x^2 + 98x + 4y^2 - 245 = 0$
- e. What can you tell about a graph of an equation of the form $ax^2 + bx + cy^2 + dy + e = 0$ by looking at the coefficients?

Lesson 9: Volume and Cavalieri's Principle

Classwork

Exercises

1. Let $R = 5$, and let $A(x)$ represent the area of a cross section for a circle at a distance x from the center of the sphere.

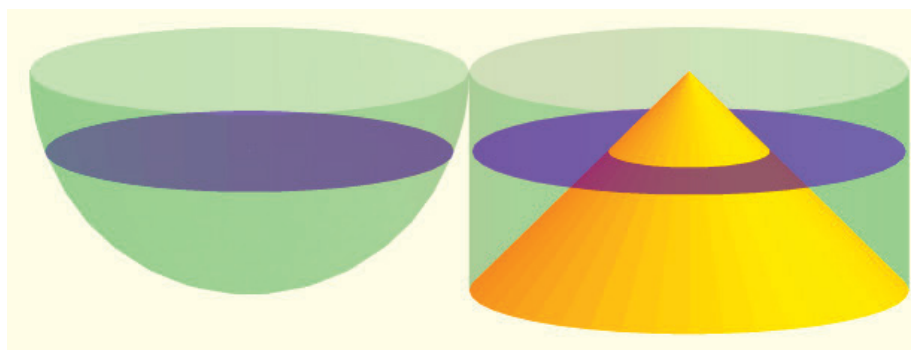


"Sphere with Cross Section" by Theneb 314 is licensed under CC BY-SA 3.0

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- a. Find $A(0)$. What is special about this particular cross section?
- b. Find $A(1)$.
- c. Find $A(3)$.

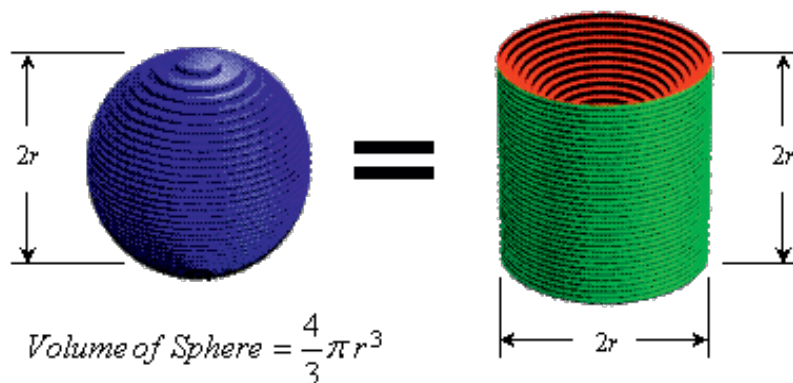
- d. Find $A(4)$.
- e. Find $A(5)$. What is special about this particular cross section?
2. Let the radius of the cylinder be $R = 5$, and let $B(x)$ represent the area of the blue ring when the slicing plane is at a distance x from the top of the cylinder.



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- a. Find $B(1)$. Compare this area with $A(1)$, the area of the corresponding slice of the sphere.
- b. Find $B(2)$. Compare this area with $A(2)$, the area of the corresponding slice of the sphere.
- c. Find $B(3)$. Compare this area with $A(3)$, the area of the corresponding slice of the sphere.

3. Explain how to derive the formula for the volume of a sphere with radius r .

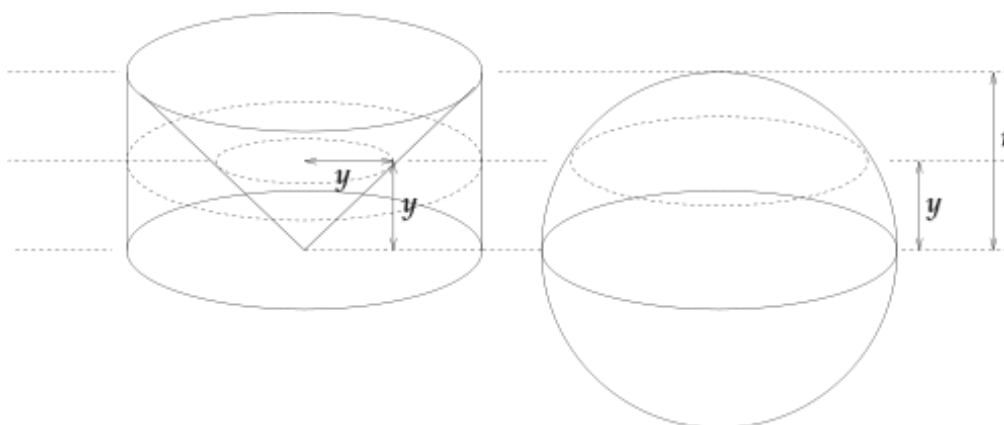


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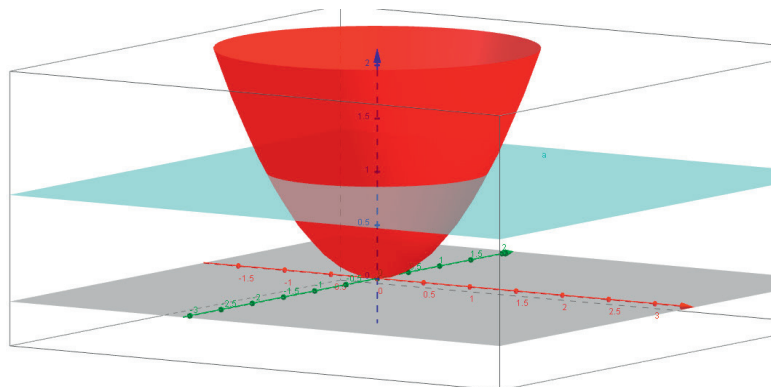
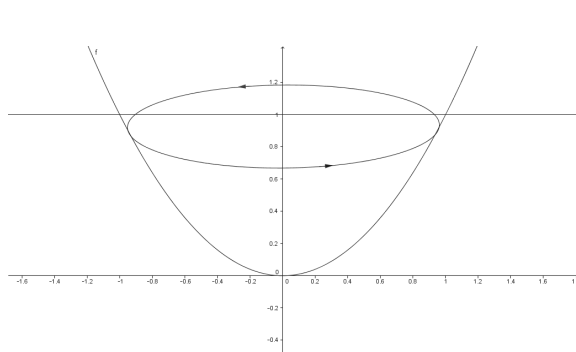
<http://www.ceemrr.com>

Problem Set

1. Consider the sphere with radius $r = 4$. Suppose that a plane passes through the sphere at a height $y = 2$ units above the center of the sphere, as shown in the figure below.

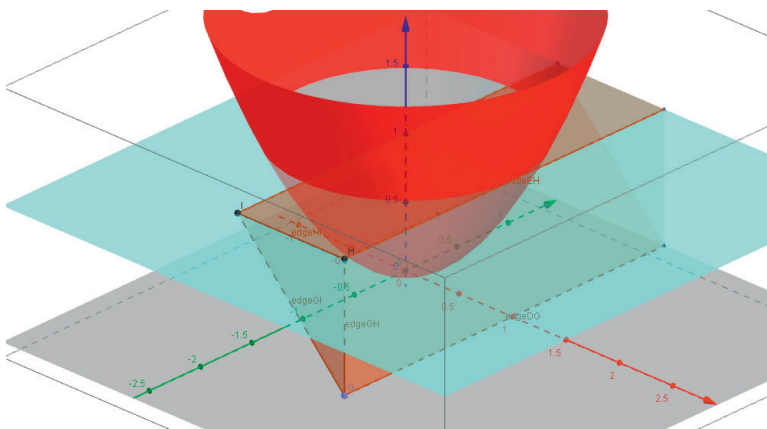


- Find the area of the cross section of the sphere.
 - Find the area of the cross section of the cylinder that lies outside of the cone.
 - Find the volume of the cylinder, the cone, and the hemisphere shown in the figure.
 - Find the volume of the sphere shown in the figure.
 - Explain using Cavalieri's principle the formula for the volume of any single solid.
2. Give an argument for why the volume of a right prism is the same as an oblique prism with the same height.
3. A *paraboloid of revolution* is a three-dimensional shape obtained by rotating a parabola around its axis. Consider the solid between a paraboloid described by the equation $y = x^2$ and the line $y = 1$.



- Cross sections perpendicular to the y -axis of this paraboloid are what shape?
- Find the area of the largest cross section of this solid, when $y = 1$.
- Find the area of the smallest cross section of this solid, when $y = 0$.

- d. Consider a right triangle prism with legs of length 1, hypotenuse of length $\sqrt{2}$, and depth π as pictured below. What shape are the cross sections of the prism perpendicular to the y -axis?



- e. Find the areas of the cross sections of the prism at $y = 1$ and $y = 0$.
- f. Verify that at $y = y_0$, the areas of the cross sections of the paraboloid and the prism are equal.
- g. Find the volume of the paraboloid between $y = 0$ and $y = 1$.
- h. Compare the volume of the paraboloid to the volume of the smallest cylinder containing it. What do you notice?
- i. Let V_{cyl} be the volume of a cylinder, V_{par} be the volume of the inscribed paraboloid, and V_{cone} be the volume of the inscribed cone. Arrange the three volumes in order from smallest to largest.
4. Consider the graph of f described by the equation $f(x) = \frac{1}{2}x^2$ for $0 \leq x \leq 10$.
- Find the area of the 10 rectangles with height $f(i)$ and width 1, for $i = 1, 2, 3, \dots, 10$.
 - What is the total area for $0 \leq x \leq 10$? That is, evaluate $\sum_{i=1}^{10} f(i) \cdot \Delta x$ for $\Delta x = 1$.
 - Draw a picture of the function and rectangles for $i = 1, 2, 3$.
 - Is your approximation an overestimate or an underestimate?
 - How could you get a better approximation of the area under the curve?
5. Consider the three-dimensional solid that has square cross sections and whose height y at position x is given by the equation $y = 2\sqrt{x}$ for $0 \leq x \leq 4$.
- Approximate the shape with four rectangular prisms of equal width. What is the height and volume of each rectangular prism? What is the total volume?
 - Approximate the shape with eight rectangular prisms of equal width. What is the height and volume of each rectangular prism? What is the total volume?
 - How much did your approximation improve? The volume of the shape is 32 cubic units. How close is your approximation from part (b)?
 - How many rectangular prisms would you need to be able to approximate the volume accurately?

Lesson 10: The Structure of Rational Expressions

Classwork

Opening Exercise

a. Add the fractions: $\frac{3}{5} + \frac{2}{7}$.

b. Subtract the fractions: $\frac{5}{2} - \frac{4}{3}$.

c. Add the expressions: $\frac{3}{x} + \frac{x}{5}$.

d. Subtract the expressions: $\frac{x}{x+2} - \frac{3}{x+1}$.

Exercises

1. Construct an argument that shows that the set of rational numbers is closed under addition. That is, if x and y are rational numbers and $w = x + y$, prove that w must also be a rational number.
2. How could you modify your argument to show that the set of rational numbers is also closed under subtraction? Discuss your response with another student.
3. Multiply the fractions: $\frac{2}{5} \cdot \frac{3}{4}$.

4. Divide the fractions: $\frac{2}{5} \div \frac{3}{4}$.

5. Multiply the expressions: $\frac{x+1}{x+2} \cdot \frac{3x}{x-4}$.

6. Divide the expressions: $\frac{x+1}{x+2} \div \frac{3x}{x-4}$.

7. Construct an argument that shows that the set of rational numbers is closed under division. That is, if x and y are rational numbers (with y nonzero) and $w = \frac{x}{y}$, prove that w must also be a rational number.
8. How could you modify your argument to show that the set of rational expressions is also closed under division by a nonzero rational expression? Discuss your response with another student.

Problem Set

1. Given $\frac{x+1}{x-2}$ and $\frac{x-1}{x^2-4}$, show that performing the following operations results in another rational expression.
 - a. Addition
 - b. Subtraction
 - c. Multiplication
 - d. Division
2. Find two rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ that produce the result $\frac{x-1}{x^2}$ when using the following operations. Answers for each type of operation may vary. Justify your answers.
 - a. Addition
 - b. Subtraction
 - c. Multiplication
 - d. Division
3. Find two rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ that produce the result $\frac{2x+2}{x^2-x}$ when using the following operations. Answers for each type of operation may vary. Justify your answers.
 - a. Addition
 - b. Subtraction
 - c. Multiplication
 - d. Division
4. Consider the rational expressions A, B and their quotient, $\frac{A}{B}$, where B is not equal to zero.
 - a. For some rational expression C , does $\frac{AC}{BC} = \frac{A}{B}$?
 - b. Let $A = \frac{x}{y} + \frac{1}{x}$ and $B = \frac{y}{x} + \frac{1}{y}$. What is the least common denominator of every term of each expression?
 - c. Find AC, BC where C is equal to your result in part (b). Then, find $\frac{AC}{BC}$. Simplify your answer.
 - d. Express each rational expression A, B as a single rational term, that is, as a division between two polynomials.
 - e. Write $\frac{A}{B}$ as a multiplication problem.
 - f. Use your answers to parts (d) and (e) to simplify $\frac{A}{B}$.
 - g. Summarize your findings. Which method do you prefer using to simplify rational expressions?

5. Simplify the following rational expressions.

a. $\frac{\frac{1}{y} - \frac{1}{x}}{\frac{x}{y} - \frac{y}{x}}$

b. $\frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x^2} - \frac{1}{y^2}}$

c. $\frac{\frac{1}{x^4} - \frac{1}{y^2}}{\frac{1}{x^4} + \frac{2}{x^2y} + \frac{1}{y^2}}$

d. $\frac{\frac{x-1}{1} - \frac{x}{1}}{\frac{1}{x-1} + \frac{1}{x}}$

6. Find A and B that make the equation true. Verify your results.

a. $\frac{A}{x+1} + \frac{B}{x-1} = \frac{2}{x^2-1}$

b. $\frac{A}{x+3} + \frac{B}{x+2} = \frac{2x-1}{x^2+5x+6}$

7. Find A , B , and C that make the equation true. Verify your result.

$$\frac{Ax+B}{x^2+1} + \frac{C}{x+2} = \frac{x-1}{(x^2+1)(x+2)}$$

Lesson 11: Rational Functions

Classwork

Opening Exercise

Factor each expression completely:

a. $9x^4 - 16x^2$

b. $2x^3 + 5x^2 - 8x - 20$

c. $x^3 + 3x^2 + 3x + 1$

d. $8x^3 - 1$

Example 1

Simplify the expression $\frac{x^2 - 5x + 6}{x - 3}$ to lowest terms, and identify the value(s) of x that must be excluded to avoid division by zero.

Exercise 1: Simplifying Rational Expressions to Lowest Terms

1. Simplify each rational expression to lowest terms, specifying the values of x that must be excluded to avoid division by zero.

a. $\frac{x^2 - 6x + 5}{x^2 - 3x - 10}$

b. $\frac{x^3 + 3x^2 + 3x + 1}{x^3 + 2x^2 + x}$

c. $\frac{x^2 - 16}{x^2 + 2x - 8}$

d. $\frac{x^2 - 3x - 10}{x^3 + 6x^2 + 12x + 8}$

e. $\frac{x^3 + 1}{x^2 + 1}$

Example 2

Let $f(x) = \frac{2x^4 + 6x^3 + 6x^2 + 2x}{3x^2 + 3x}$. Simplify the rational expression $\frac{2x^4 + 6x^3 + 6x^2 + 2x}{3x^2 + 3x}$ to lowest terms, and use the simplified form to express the rule of f . Be sure to indicate any restrictions on the domain.

Exercise 2

2. Determine the domain of each rational function, and express the rule for each function in an equivalent form in lowest terms.

a. $f(x) = \frac{(x+2)^2(x-3)(x+1)}{(x+2)(x+1)}$

b. $f(x) = \frac{x^2 - 6x + 9}{x - 3}$

c. $f(x) = \frac{3x^3 - 75x}{x^3 + 15x^2 + 75x + 125}$

Problem Set

1. For each pair of functions f and g , find the domain of f and the domain of g . Indicate whether f and g are the same function.

- $f(x) = \frac{x^2}{x}$, $g(x) = x$
- $f(x) = \frac{x}{x}$, $g(x) = 1$
- $f(x) = \frac{2x^2 + 6x + 8}{2}$, $g(x) = x^2 + 6x + 8$
- $f(x) = \frac{x^2 + 3x + 2}{x + 2}$, $g(x) = x + 1$
- $f(x) = \frac{x + 2}{x^2 + 3x + 2}$, $g(x) = \frac{1}{x + 1}$
- $f(x) = \frac{x^4 - 1}{x^2 - 1}$, $g(x) = x^2 + 1$
- $f(x) = \frac{x^4 - 1}{x^2 + 1}$, $g(x) = x^2 - 1$
- $f(x) = \frac{x^4 - x}{x^2 + x}$, $g(x) = \frac{x^3 - 1}{x + 1}$
- $f(x) = \frac{x^4 + x^3 + x^2}{x^2 + x + 1}$, $g(x) = x^2$

2. Determine the domain of each rational function, and express the rule for each function in an equivalent form in lowest terms.

- $f(x) = \frac{x^4}{x^2}$
- $f(x) = \frac{3x + 3}{15x - 6}$
- $f(x) = \frac{x^2 - x - 2}{x^2 + x}$
- $f(x) = \frac{8x^2 + 2x - 15}{4x^2 - 4x - 15}$
- $f(x) = \frac{2x^3 - 3x^2 - 2x + 3}{x^3 - x}$
- $f(x) = \frac{3x^3 + x^2 + 3x + 1}{x^3 + x}$

3. For each pair of functions below, calculate $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$, and $\frac{f(x)}{g(x)}$. Indicate restrictions on the domain of the resulting functions.

- $f(x) = \frac{2}{x}$, $g(x) = \frac{x}{x + 2}$
- $f(x) = \frac{3}{x + 1}$, $g(x) = \frac{x}{x^3 + 1}$

Lesson 12: End Behavior of Rational Functions

Classwork

Opening Exercise

Analyze the end behavior of each function below. Then, choose one of the functions, and explain how you determined the end behavior.

a. $f(x) = x^4$

b. $g(x) = -x^4$

c. $h(x) = x^3$

d. $k(x) = -x^3$

Exercises

Determine the end behavior of each rational function below.

1. $f(x) = \frac{7x^5 - 3x + 1}{4x^3 + 2}$

2. $f(x) = \frac{7x^3 - 3x + 1}{4x^3 + 2}$

3. $f(x) = \frac{7x^3 + 2}{4x^5 - 3x + 1}$

Problem Set

1. Analyze the end behavior of both functions.

a. $f(x) = x, g(x) = \frac{1}{x}$
b. $f(x) = x^3, g(x) = \frac{1}{x^3}$
c. $f(x) = x^2, g(x) = \frac{1}{x^2}$
d. $f(x) = x^4, g(x) = \frac{1}{x^4}$
e. $f(x) = x - 1, g(x) = \frac{1}{x - 1}$
f. $f(x) = x + 2, g(x) = \frac{1}{x + 2}$
g. $f(x) = x^2 - 4, g(x) = \frac{1}{x^2 - 4}$

2. For the following functions, determine the end behavior. Confirm your answer with a table of values.

a. $f(x) = \frac{3x - 6}{x + 2}$
b. $f(x) = \frac{5x + 1}{x^2 - x - 6}$
c. $f(x) = \frac{x^3 - 8}{x^2 - 4}$
d. $f(x) = \frac{x^3 - 1}{x^4 - 1}$
e. $f(x) = \frac{(2x + 1)^3}{(x^2 - x)^2}$

3. For the following functions, determine the end behavior.

a. $f(x) = \frac{5x^6 - 3x^3 + x - 2}{5x^4 - 3x^3 + x - 2}$
b. $f(x) = \frac{5x^4 - 3x^3 + x - 2}{5x^6 - 3x^3 + x - 2}$
c. $f(x) = \frac{5x^4 - 3x^3 + x - 2}{5x^4 - 3x^3 + x - 2}$
d. $f(x) = \frac{\sqrt{2}x^2 + x + 1}{3x + 1}$
e. $f(x) = \frac{4x^2 - 3x - 7}{2x^3 + x - 2}$

4. Determine the end behavior of each function.

a. $f(x) = \frac{\sin(x)}{x}$

b. $f(x) = \frac{\cos(x)}{x}$

c. $f(x) = \frac{2^x}{x}$

d. $f(x) = \frac{x}{2^x}$

e. $f(x) = \frac{4}{1 + e^{-x}}$

f. $f(x) = \frac{10}{1 + e^{-x}}$

5. Consider the functions $f(x) = x!$ and $g(x) = x^5$ for natural numbers x .

a. What are the values of $f(x)$ and $g(x)$ for $x = 5, 10, 15, 20, 25$?

b. What is the end behavior of $f(x)$ as $x \rightarrow \infty$?

c. What is the end behavior of $g(x)$ as $x \rightarrow \infty$?

d. Make an argument for the end behavior of $\frac{f(x)}{g(x)}$ as $x \rightarrow \infty$.

e. Make an argument for the end behavior of $\frac{g(x)}{f(x)}$ as $x \rightarrow \infty$.

6. Determine the end behavior of the functions.

a. $f(x) = \frac{x}{x^2}, g(x) = \frac{1}{x}$

b. $f(x) = \frac{x+1}{x^2-1}, g(x) = \frac{1}{x-1}$

c. $f(x) = \frac{x-2}{x^2-x-2}, g(x) = \frac{1}{x+1}$

d. $f(x) = \frac{x^3-1}{x-1}, g(x) = x^2 + x + 1$

7. Use a graphing utility to graph the following functions f and g . Explain why they have the same graphs. Determine the end behavior of the functions and whether the graphs have any horizontal asymptotes.

a. $f(x) = \frac{x+1}{x-1}, g(x) = 1 + \frac{2}{x-1}$

b. $f(x) = \frac{-2x+1}{x+1}, g(x) = \frac{3}{x+1} - 2$

Lesson 13: Horizontal and Vertical Asymptotes of Graphs of Rational Functions

Classwork

Opening Exercise

Determine the end behavior of each rational function below. Graph each function on the graphing calculator, and explain how the graph supports your analysis of the end behavior.

a. $f(x) = \frac{x^2 - 3}{x^3}$

b. $f(x) = \frac{x^2 - 3}{x^2 + 1}$

c. $f(x) = \frac{x^3}{x^2 - 3}$

Example

Consider the rational function $f(x) = \frac{2x-1}{x-4}$.

- State the domain of f .
- Determine the end behavior of f .
- State the horizontal asymptote of the graph of $y = f(x)$.
- Graph the function on the graphing calculator, and make a sketch on your paper.

Exercises

State the domain and end behavior of each rational function. Identify all horizontal and vertical asymptotes on the graph of each rational function. Then, verify your answer by graphing the function on the graphing calculator.

1. $f(x) = \frac{-x+6}{2x+3}$

2. $f(x) = \frac{3x-6}{x}$

3. $f(x) = \frac{3}{x^2-25}$

$$4. \quad f(x) = \frac{x^2 - 2}{x^2 + 2x - 3}$$

$$5. \quad f(x) = \frac{x^2 - 5x - 4}{x + 1}$$

$$6. \quad f(x) = \frac{5x}{x^2 + 9}$$

Write an equation for a rational function whose graph has the given characteristic. Graph your function on the graphing calculator to verify.

7. A horizontal asymptote of $y = 2$ and a vertical asymptote of $x = -2$

8. A vertical asymptote of $x = 6$ and no horizontal asymptote

9. A horizontal asymptote of $y = 6$ and no vertical asymptote

Lesson Summary

- Let a be a real number. The line given by $x = a$ is a *vertical asymptote* of the graph of $y = f(x)$ if at least one of the following statements is true:
 - As $x \rightarrow a$, $f(x) \rightarrow \infty$.
 - As $x \rightarrow a$, $f(x) \rightarrow -\infty$.
- Let L be a real number. The line given by $y = L$ is a *horizontal asymptote* of the graph of $y = f(x)$ if at least one of the following statements is true:
 - As $x \rightarrow \infty$, $f(x) \rightarrow L$.
 - As $x \rightarrow -\infty$, $f(x) \rightarrow L$.

Problem Set

1. State the domain of each rational function. Identify all horizontal and vertical asymptotes on the graph of each rational function.

a. $y = \frac{3}{x^3 - 1}$

b. $y = \frac{2x + 2}{x - 1}$

c. $y = \frac{5x^2 - 7x + 12}{x^3}$

d. $y = \frac{3x^6 - 2x^3 + 1}{16 - 9x^6}$

e. $f(x) = \frac{6 - 4x}{x + 5}$

f. $f(x) = \frac{4}{x^2 - 4}$

2. Sketch the graph of each function in Exercise 1 with asymptotes and excluded values from the domain drawn on the graph.

3. Factor out the highest power of x in each of the following, and cancel common factors if you can. Assume x is nonzero.

a. $y = \frac{x^3 + 3x - 4}{3x^3 - 4x^2 + 2x - 5}$

b. $y = \frac{x^3 - x^2 - 6x}{x^3 + 5x^2 + 6x}$

c. $y = \frac{2x^4 - 3x + 1}{5x^3 - 8x - 1}$

d. $y = -\frac{9x^5 - 8x^4 + 3x + 72}{7x^5 + 8x^4 + 8x^3 + 9x^2 + 10x}$

e. $y = \frac{3x}{4x^2 + 1}$

4. Describe the end behavior of each function in Exercise 3.
5. Using the equations that you wrote in Exercise 3, make some generalizations about how to quickly determine the end behavior of a rational function.
6. Describe how you may be able to use the end behavior of the graphs of rational functions, along with the excluded values from the domain and the equations of any asymptotes, to graph a rational function without technology.

Lesson 14: Graphing Rational Functions

Classwork

Opening Exercise

State the domain of each of the following functions. Then, determine whether or not the excluded value(s) of x are vertical asymptotes on the graph of the function. Give a reason for your answer.

a. $f(x) = \frac{x^2 - 3x + 2}{x - 2}$

b. $f(x) = \frac{x^2 + 3x + 2}{x - 2}$

Example 1

Sketch the graph of the rational function $f(x) = \frac{2x^2 - x}{x^2 - 16}$ showing all the key features of the graph. Label the key features on your graph.

Example 2

Graph the function $f(x) = \frac{x^2 + 5x - 6}{x + 1}$ showing all the key features.

Exercises

Sketch the graph of each rational function showing all the key features. Verify your graph by graphing the function on the graphing calculator.

1. $f(x) = \frac{4x - 6}{2x + 5}$

2. $f(x) = \frac{(3x - 6)(x - 4)}{x(x - 4)}$

3. $f(x) = \frac{3x - 2x^2}{x - 2}$

4. $f(x) = \frac{x - 2}{3x - 2x^2}$

5. $f(x) = \frac{x}{x^2 - 9}$

6. $f(x) = \frac{x^2}{x^2 - 9}$

7. $f(x) = \frac{x^2 - 9}{x}$

8. $f(x) = \frac{x^2 - 9x}{x}$

9. $f(x) = \frac{x^3 - 8}{x - 2}$

10. $f(x) = \frac{x^3 - 8}{x - 1}$

Problem Set

1. List all of the key features of each rational function and its graph, and then sketch the graph showing the key features.
 - a. $y = \frac{x}{x-1}$
 - b. $y = \frac{x^2 - 7x + 6}{x^2 - 36}$
 - c. $y = \frac{x^3 - 3x^2 - 10x}{x^2 + 8x - 65}$
 - d. $y = \frac{3x}{x^2 - 1}$
2. Graph $y = \frac{1}{x^2}$ and $y = \frac{1}{x}$. Compare and contrast the two graphs.

Extension:

3. Consider the function $f(x) = \frac{x^3 + 1}{x}$.
 - a. Use the distributive property to rewrite f as the sum of two rational functions g and h .
 - b. What is the end behavior of g ? What is the end behavior of h ?
 - c. Graph $y = f(x)$ and $y = x^2$ on the same set of axes. What do you notice?
 - d. Summarize what you have discovered in parts (b) and (c).
4. Number theory is a branch of mathematics devoted primarily to the study of integers. Some discoveries in number theory involve numbers that are impossibly large such as Skewes' numbers and Graham's number. One Skewes' number is approximately $e^{(e^{79})}$, and Graham's number is so large that to even write it requires 64 lines of writing with a new operation (one that can be thought of as the shortcut for repeated exponentiation). In fact, both of these numbers are so large that the decimal representation of the numbers would be larger than the known universe and dwarf popular large numbers such as googol and googolplex (10^{100} and $10^{(10^{100})}$, respectively). These large numbers, although nearly impossible to comprehend, are still not at the "end" of the real numbers, which have no end. Consider the function $f(x) = x^2 - 10^{100}$.
 - a. Consider only positive values of x ; how long until $f(x) > 0$?
 - b. If your answer to part (a) represented seconds, how many billions of years would it take for $f(x) > 0$? (Note: One billion years is approximately 3.15×10^{16} seconds.) How close is this to the estimated geological age of the earth (4.54 billion years)?

- c. Number theorists frequently only concern themselves with the term of a function that has the most influence as $x \rightarrow \infty$. Let $f(x) = x^3 + 10x^2 + 100x + 1000$, and answer the following questions.

i. Fill out the following table:

x	$f(x)$	x^3	$\frac{x^3}{f(x)}$	$10x^2$	$\frac{10x^2}{f(x)}$	$100x$	$\frac{100x}{f(x)}$	$\frac{1000}{f(x)}$
0								
10								
100								
1,000								

- ii. As $x \rightarrow \infty$, which term of f dominates the value of the function?
- iii. Find $g(x) = \frac{f(x)}{x}$. Which term dominates g as $x \rightarrow \infty$?
- d. Consider the formula for a general polynomial, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ for real numbers a_i , $0 \leq i \leq n$. Which single term dominates the value of f as $x \rightarrow \infty$?

Lesson 15: Transforming Rational Functions

Classwork

Exploratory Challenge/Exercises 1–2

1. Sketch the general shape of the graph of the function $f(x) = \frac{1}{x^n}$ for $n > 0$ when n is an odd number.
2. Sketch the general shape of the graph of the function $f(x) = \frac{1}{x^n}$ for $n > 0$ when n is an even number.

Exercises 3–5

3. Sketch the graph of the function $f(x) = \frac{1}{x}$. Then, use the graph of f to sketch each transformation of f showing the vertical and horizontal asymptotes.

a. $g(x) = \frac{1}{x-2}$

b. $h(x) = -\frac{1}{x} + 3$

c. $k(x) = \frac{2}{x+3} - 5$

4. Use your results from Exercise 3 to make some general statements about graphs of functions in the form $f(x) = a + \frac{b}{x-c}$. Describe the effect that changing each parameter a , b , and c has on the graph of f .

5. Sketch the graph of the function $f(x) = \frac{1}{x^2}$. Then, use the graph of f to sketch each transformation of f showing the vertical and horizontal asymptotes.

a. $g(x) = -\frac{3}{(x+1)^2}$

b. $h(x) = \frac{1}{(x-1)^2} + 4$

Example 1

Graph the function $f(x) = \frac{x+2}{x-3}$ using transformations of the graph of $y = \frac{1}{x}$.

Exercises 6–13

Sketch each function by using transformations of the graph of $y = \frac{1}{x}$ or the graph of $y = \frac{1}{x^2}$. Explain the transformations that are evident in each example.

6. $f(x) = \frac{x-7}{x-5}$

7. $f(x) = \frac{2x+6}{x+1}$

8. $f(x) = \frac{2x^2 - 1}{x^2}$

9. $f(x) = \frac{1 + 4x^3}{x^3}$

10. $f(x) = \frac{x^2 - 2x + 3}{(x - 1)^2}$

11. $f(x) = \frac{2x^2 + 12x + 13}{(x + 3)^2}$

12. $f(x) = \frac{x+4}{x^2-16}$

13. $f(x) = \frac{x}{x^3-4x^2+4x}$

Problem Set

- Write each function so that it appears to be a transformation of $y = \frac{1}{x^n}$. Then, explain how the graph of each function relates to the graph of $y = \frac{1}{x^n}$.
 - $y = \frac{5x - 8}{x + 2}$
 - $y = \frac{2x^3 - 4}{x^3}$
 - $y = \frac{x^2 - 4x + 8}{(x - 2)^2}$
 - $y = \frac{3x - 12}{x^2 - 16}$
 - $y = \frac{2x^2 + 16x + 25}{x^2 + 8x + 16}$
- For each function in Problem 1, state how the horizontal and vertical asymptotes are affected from the original graph of $y = \frac{1}{x^n}$.
- Sketch a picture of the graph of each function in Problem 1.
- What are some indicators whether or not a rational function can be expressed as a transformation of $y = \frac{1}{x^n}$?
- Write an equation for a function whose graph is a transformation of the graph $y = \frac{1}{x}$. The graph has been shifted right 2 units, stretched vertically by a factor of 2, and shifted down 3 units.

Lesson 16: Function Composition

Classwork

Example 1

Consider the tables from the opening scenario.

Depth of Free Diver During Descent									
s time of descent, in seconds	20	40	60	80	100	120	140	160	180
d depth of diver, in meters	15	32	44	65	79	90	106	120	133

Atmospheric Pressure and Ocean Depth									
d depth of diver, in meters	10	20	30	40	50	60	70	80	90
p pressure on diver, in atm	2	3	4	5	6	7	8	9	10

- Do the tables appear to represent functions? If so, define the function represented in each table using a verbal description.
- What are the domain and range of the functions?

- c. Let's define the function in the first table as $d = f(s)$ and the function in the second table as $p = g(d)$. Use function notation to represent each output, and use the appropriate table to find its value.
- Depth of the diver at 80 seconds
 - Pressure on the diver at a depth of 60 meters
- d. Explain how we could determine the pressure applied to a diver after 120 seconds of descent.
- e. Use function notation to represent part (d), and use the tables to evaluate the function.
- f. Describe the output from part (e) in context.

Example 2

Consider these functions:

f : Animals \rightarrow Counting numbers

Assign to each animal the number of legs it has.

g : People \rightarrow Animals

Assign to each person his favorite animal.

Describe which composite functions are defined. If defined, describe the action of each composite function.

- a. $f \circ g$

b. $f \circ f$

c. $g \circ f$

d. $f \circ g \circ g$

Exercises

1. Let $f(x) = x^2$ and $g(x) = x + 5$. Write an expression that represents each composition:

a. $g(f(4))$

b. $f(g(4))$

c. $(f \circ g)(x)$

d. $(f \circ g)(\sqrt{x+5})$

2. Suppose a sports medicine specialist is investigating the atmospheric pressure placed on competitive free divers during their descent. The following table shows the depth, d , in meters of a free diver s seconds into his descent. The depth of the diver is a function of the number of seconds the free diver has descended, $d = f(s)$.

s time of descent, s	10	35	55	70	95	115	138	160	175
d depth, m	8.1	28	45	55	76.0	91.5	110	130	145

The pressure, in atmospheres, felt on a free diver, d , is a function of his depth, $p = g(d)$.

d depth, m	25	35	55	75	95	115	135	155	175
p pressure, atm	2.4	3.5	5.5	7.6	9.6	11.5	13.7	15.5	17.6

- a. How can the researcher use function composition to examine the relationship between the time a diver spends descending and the pressure he experiences? Use function notation to explain your response.

- b. Explain the meaning of $g(f(0))$ in context.

- c. Use the charts to approximate these values, if possible. Explain your answers in context.

i. $g(f(70))$

ii. $g(f(160))$

Problem Set

1. Determine whether each rule described represents a function. If the rule represents a function, write the rule using function notation, and describe the domain and range.
 - a. Assign to each person her age in years.
 - b. Assign to each person his height in centimeters.
 - c. Assign to each piece of merchandise in a store a bar code.
 - d. Assign each deli customer a numbered ticket.
 - e. Assign a woman to her child.
 - f. Assign to each number its first digit.
 - g. Assign each person to the city where he was born.
2. Let $L: \text{Animal} \rightarrow \text{Counting numbers}$
Assign each animal to its number of legs.
 $F: \text{People} \rightarrow \text{Animals}$
Assign to each person to his favorite animal.
 $N: \text{People} \rightarrow \text{Alphabet}$
Assign each person to the first letter of her name.
 $A: \text{Alphabet} \rightarrow \text{Counting numbers}$
Assign each letter to the corresponding number 1–26.
 $S: \text{Counting numbers} \rightarrow \text{Counting numbers}$
Assign each number its square.

Which of the following compositions are defined? For those that are, describe the effect of the composite function.

- a. $L \circ F$
- b. $N \circ L$
- c. $A \circ L$
- d. $A \circ N$
- e. $N \circ A$
- f. $F \circ L$
- g. $S \circ L \circ F$
- h. $A \circ A \circ N$

3. Let $f(x) = x^2 - x$, $g(x) = 1 - x$.

- $f \circ g$
- $g \circ f$
- $g \circ g$
- $f \circ f$
- $f(g(2))$
- $g(f(-1))$

4. Let $f(x) = x^2$, $g(x) = x + 3$.

- $g(f(5))$
- $f(g(5))$
- $f(g(x))$
- $g(f(x))$
- $g(f(\sqrt{x+3}))$

5. Let $f(x) = x^3$, $g(x) = \sqrt[3]{x}$.

- $f \circ g$
- $g \circ f$
- $f(g(8))$
- $g(f(2))$
- $f(g(-8))$
- $g(f(-2))$

6. Let $f(x) = x^2$, $g(x) = \sqrt{x} + 3$.

- Show that $(f(x+3)) = |x+3| + 3$.
- Does $(x) = |x+3| + 3 = (x) = |x| + 6$? Graph them on the same coordinate plane.

7. Given the chart below, find the following:

	-6	0	2	4
$f(x)$	4	-6	0	2
$g(x)$	2	4	-6	0
$h(x)$	0	2	4	-6
$k(x)$	1	4	0	3

- $f(g(0))$
- $g(k(2))$
- $k(g(-6))$
- $g(h(4))$

- e. $g(k(4))$
 f. $(f \circ g \circ h)(2)$
 g. $(f \circ f \circ f)(0)$
 h. $(f \circ g \circ h \circ g)(2)$

8. Suppose a flu virus is spreading in a community. The following table shows the number of people, n , who have the virus d days after the initial outbreak. The number of people who have the virus is a function of the number of days, $n = f(d)$.

d days	0	1	4	8	12	16	20
$n = f(d)$ number of people infected	2	4	14	32	64	50	32

There is only one pharmacy in the community. As the number of people who have the virus increases, the number of boxes of cough drops, b , sold also increases. The number of boxes of cough drops sold on a given day is a function of the number of people who have the virus, $b = g(n)$, on that day.

n number of people infected	0	2	4	9	14	20	28	32	44	48	50	60	64
$b = g(n)$ number of boxes of cough drops sold	1	5	14	16	22	30	42	58	74	86	102	124	136

- a. Find $g(f(1))$, and state the meaning of the value in the context of the flu epidemic. Include units in your answer.
 b. Fill in the chart below using the fact that $b = g(f(d))$.

d (days)	0	1	4	8	12	16	20
b number of boxes of cough drops sold							

- c. For each of the following expressions, interpret its meaning in the context of the problem, and if possible, give an approximation of its value.
- $g(f(4))$
 - $g(f(16))$
 - $f(g(9))$

Lesson 17: Solving Problems by Function Composition

Classwork

Example 1

Find the domain and range for the following functions:

a. $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$

b. $g: \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) = \sqrt{x - 2}$

c. $f(g(x))$

d. $g(f(x)) = g(x^2) = \sqrt{x^2 - 2}$

Exercise 1

1. Find the domain and range for the following functions:

a. $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x + 2$

b. $g: \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) = \sqrt{x - 1}$

c. $f(g(x))$

d. $g(f(x))$

Example 2

According to the Global Wind Energy Council, a wind turbine can generate about 16,400 kWh of power each day. According to the Alternative Fuels Data Center, an average electric car can travel approximately 100 miles on 34 kWh of energy. An environmental nonprofit organization is interested in analyzing how wind power could offset the energy use of electric vehicles.

- Write a function that represents the relationship between the number of wind turbines operating in a wind farm and the amount of energy they generate per day (in kilowatt-hours). Define the input and output.
- Write a function that represents the relationship between the energy expended by an electric car (in kilowatt-hours) and the number of miles driven.
- Write a function that could be used to determine the number of miles that an electric car could drive based on the number of wind turbines operating daily at a wind farm. Interpret this function in context.

- d. Determine an appropriate domain and range for part (c). Explain why your domain and range are reasonable in this context.

- e. How many miles of driving could be generated daily by 20 wind turbines in a day?

Exercises 2–3

2. A product safety commission is studying the effect of rapid temperature changes on the equipment of skydivers as they descend. The commission has collected data on a typical skydiver during the part of the dive when she has reached terminal velocity (maximum speed) to the time the parachute is released. They know that the terminal velocity of a diver is approximately 56 m/s and that, given the altitude of skydivers at terminal velocity, the temperature decreases at an average rate of $6.4 \frac{^{\circ}\text{C}}{\text{km}}$.
- a. Write a function that represents the altitude of a skydiver experiencing terminal velocity if she reaches this speed at a height of 3,000 m.

- b. Write a function that represents the relationship between the altitude of the skydiver and the temperature if the temperature at 3,000 m is 5.8°C .
- c. Write a function that could be used to determine the temperature, in degrees Celsius, of the air surrounding a skydiver based on the time she has spent descending at terminal velocity. Interpret the equation in context.

- d. Determine an appropriate domain and range for part (c).
- e. How long would it take a skydiver to reach an altitude where the temperature is 8°C ?
3. A department store manager is planning to move some cement spheres that have served as traffic barriers for the front of her store. She is trying to determine the relationship between the mass of the spheres and their diameter in meters. She knows that the density of the cement is approximately $2,500 \text{ kg/m}^3$.
- a. Write a function that represents the relationship between the volume of a sphere and its diameter. Explain how you determined the equation.
- b. Write a function that represents the relationship between the mass and the volume of the sphere. Explain how you determined the function.

- c. Write a function that could be used to determine the mass of one of the cement spheres based on its diameter. Interpret the equation in context.
- d. Determine an appropriate domain and range for part (c).
- e. What is the approximate mass of a sphere with a diameter of 0.9 m?

Problem Set

1. Find the domain and range of the following functions:
 - a. $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = -x^2 + 2$
 - b. $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \frac{1}{x+1}$
 - c. $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \sqrt{4-x}$
 - d. $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = |x|$
 - e. $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 2^{x+2}$
2. Given $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$, for the following, find $f(g(x))$ and $g(f(x))$, and state the domain.
 - a. $f(x) = x^2 - x$, $g(x) = x - 1$
 - b. $f(x) = x^2 - x$, $g(x) = \sqrt{x-2}$
 - c. $f(x) = x^2$, $g(x) = \frac{1}{x-1}$
 - d. $f(x) = \frac{1}{x+2}$, $g(x) = \frac{1}{x-1}$
 - e. $f(x) = x - 1$, $g(x) = \log_2(x + 3)$
3. A company has developed a new highly efficient solar panel. Each panel can produce 0.75 MW of electricity each day. According to the Los Angeles power authority, all the traffic lights in the city draw 0.5 MW of power per day.
 - a. Write a function that represents the relationship between the number of solar panels installed and the amount of energy generated per day (in MWh). Define the input and output.
 - b. Write a function that represents the relationship between the number of days and the energy in megawatts consumed by the traffic lights. (How many days can one megawatt provide?)
 - c. Write a function that could be used to determine the number of days that the traffic lights stay on based on the number of panels installed.
 - d. Determine an appropriate domain and range for part (c).
 - e. How many days can 20 panels power all the lights?
4. A water delivery person is trying to determine the relationship between the mass of the cylindrical containers he delivers and their diameter in centimeters. The density of the bottles is 1 g/cm^3 . The height of each bottle is approximately 60 cm.
 - a. Write a function that represents the relationship between the volume of the cylinder and its diameter.
 - b. Write a function that represents the relationship between the mass and volume of the cylinder.
 - c. Write a function that could be used to determine the mass of one cylinder based on its diameter. Interpret the equation in context.
 - d. Determine an appropriate domain and range for part (c).
 - e. What is the approximate mass of a cylinder with a diameter of 30 cm?

5. A gold mining company is mining gold in Northern California. Each mining cart carries an average 500 kg of dirt and rocks that contain gold from the tunnel. For each 2 metric tons of material (dirt and rocks), the company can extract an average of 10 g of gold. The average wholesale gold price is \$20/g.
- Write a function that represents the relationship between the mass of the material mined in metric tons and the number of carts. Define the input and output.
 - Write a function that represents the relationship between the amount of gold and the materials. Define the input and output.
 - Write a function that could be used to determine the mass of gold in metric tons as a function of the number of carts coming out from the mine.
 - Determine an appropriate domain and range for part (c).
 - Write a function that could be used to determine the amount of money the gold is worth in dollars and the amount of gold extracted in metric tons.
 - How much gold can 40,000 carts of material produce?
 - How much, in dollars, can 40,000 carts of material produce?
6. Bob operates hot air balloon rides for tourists at the beach. The hot air balloon rises, on average, at 100 feet per minute. At sea level, the atmospheric pressure, measured in inches of mercury, is 29.9 inHg. Using a barometric meter, Bob notices that the pressure decreases by 0.5 inHg for each 500 feet the balloon rises.
- Write a function that represents the relationship between the height of the hot air balloon and the time spent to reach that height.
 - Write a function that represents the relationship between the height of the hot air balloon and the atmospheric pressure being applied to the balloon.
 - Write a function that could be used to determine the pressure on the hot air balloon based on the time it spends rising.
 - Determine an appropriate domain and range for part (c).
 - What is the reading on the barometer 10 minutes after the hot air balloon has left the ground?

Lesson 18: Inverse Functions

Classwork

Businesses must track the value of their assets over time. When a business buys equipment, the value of the equipment is reduced over time. For example, electric companies provide trucks for their workers when they go out into the field to repair electrical lines. These trucks lose value over time but are still part of the business assets. For accounting purposes, many businesses use a technique called *straight-line depreciation* to calculate the value of equipment over time.

Exercises

Suppose ABC Electric purchases a new work truck for \$34,500. They estimate that the truck's value will depreciate to \$0 over 15 years. The table below shows the value $v(t)$ of the truck in thousands of dollars depreciated over time t in months using a straight-line depreciation method.

t	0	12	24	36	48	60	72	84	96
$v(t)$	34.5	32.2	29.9	27.6	25.3	23.0	20.7	18.4	16.1

1. Does the function v appear to be a linear function? Explain your reasoning.
2. What is an appropriate domain and range for v in this situation?
3. Write a formula for v in terms of t , the months since the truck was purchased.
4. What will the truck be worth after 30 months? 40 months? 50 months?

5. When will the truck be valued at \$30,000? \$20,000? \$10,000?
6. Construct a table that shows the time of depreciation, $t(v)$, in months as a function of the value of the truck, v , in thousands of dollars.
7. Does the function t appear to be a linear function? Explain your reasoning.
8. What is an appropriate domain and range for t in this situation?
9. Write a formula for t in terms of the value of the truck, v , since it was purchased.

10. Explain how you can create the formula for t using the formula for v from Exercise 5.
11. Sketch a graph of the equations $y = v(t)$ and $y = t(v)$ in the Cartesian plane. How do their graphs compare?

12. What is the meaning of the intersection point of the graphs of the two equations?
13. Add the graph of $y = x$ to your work in Exercise 11. Describe the relationship between the graphs of $y = v(t)$, $y = t(v)$, and $y = x$.
14. ABC Electric uses this formula, $f(x) = 750 - 10x$, to depreciate computers, where f is the value of a computer and x is the number of months since its purchase.
- Calculate $f(36)$. What is the meaning of $f(36)$?
 - What is the meaning of b in $f(b) = 60$? What is the value of b ?
 - Write a formula for f^{-1} , and explain what it means in this situation.
 - When will the depreciated value of a computer be less than \$400?
 - What is the meaning of c in $f^{-1}(c) = 60$? What is the value of c ?

15. Find the inverses of the following functions:

a. $f(x) = \frac{2}{3}x - 10$

b. $g(x) = 2(x + 4)^3$

c. $h(x) = \frac{1}{x-2}, x \neq 2$

Lesson Summary

- **INVERTIBLE FUNCTION:** Let f be a function whose domain is the set X and whose image (range) is the set Y . Then, f is *invertible* if there exists a function g with domain Y and image (range) X such that f and g satisfy the property:

For all x in X and y in Y , $f(x) = y$ if and only if $g(y) = x$.

The function g is called the *inverse* of f .

- If two functions whose domain and range are a subset of the real numbers are inverses, then their graphs are reflections of each other across the diagonal line given by $y = x$ in the Cartesian plane.
- If f and g are inverses of each other, then:
 - The domain of f is the same set as the range of g .
 - The range of f is the same set as the domain of g .
- The inverse of a function f is denoted f^{-1} .
- In general, to find the formula for an inverse function g of a given function f :
 - Write $y = f(x)$ using the formula for f .
 - Interchange the symbols x and y to get $x = f(y)$.
 - Solve the equation for y to write y as an expression in x .
 - Then, the formula for f^{-1} is the expression in x found in the previous step.

Problem Set

1. For each of the following, write the inverse of the function given.

a. $f = \{(1, 3), (2, 15), (3, 8), (4, -2), (5, 0)\}$

b. $g = \{(0, 5), (2, 10), (4, 15), (6, 20)\}$

c. $h = \{(1, 5), (2, 25), (3, 125), (4, 625)\}$

d.

x	1	2	3	4
$f(x)$	3	12	27	48

e.

x	-1	0	1	2
$g(x)$	3	6	12	24

f.

x	1	10	100	1,000
$h(x)$	0	1	2	3

g. $y = 2x$

h. $y = \frac{1}{3}x$

i. $y = x - 3$

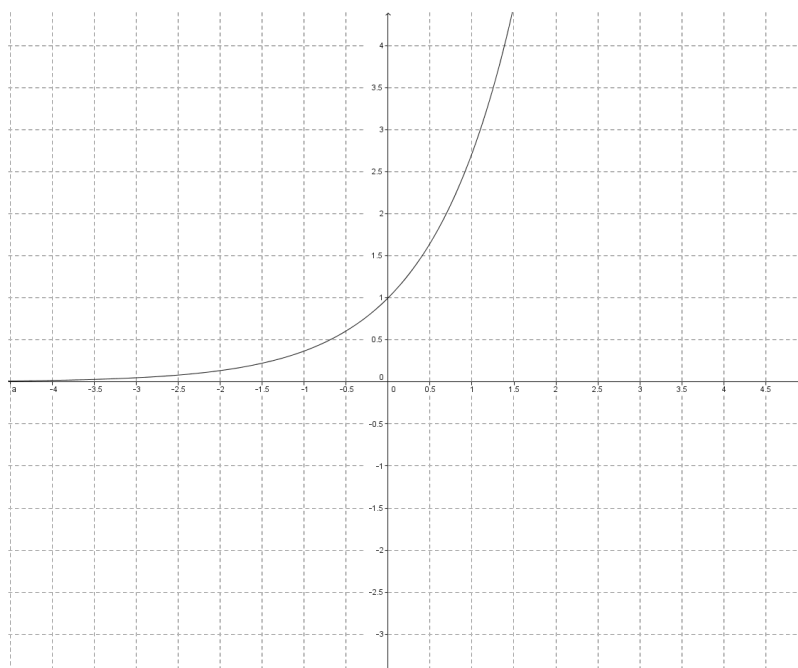
j. $y = -\frac{2}{3}x + 5$

- k. $2x - 5y = 1$
- l. $-3x + 7y = 14$
- m. $y = \frac{1}{3}(x - 9)^3$
- n. $y = \frac{5}{3x-4}, x \neq \frac{4}{3}$
- o. $y = 2x^7 + 1$
- p. $y = \sqrt[5]{x}$
- q. $y = \frac{x+1}{x-1}, x \neq 1$

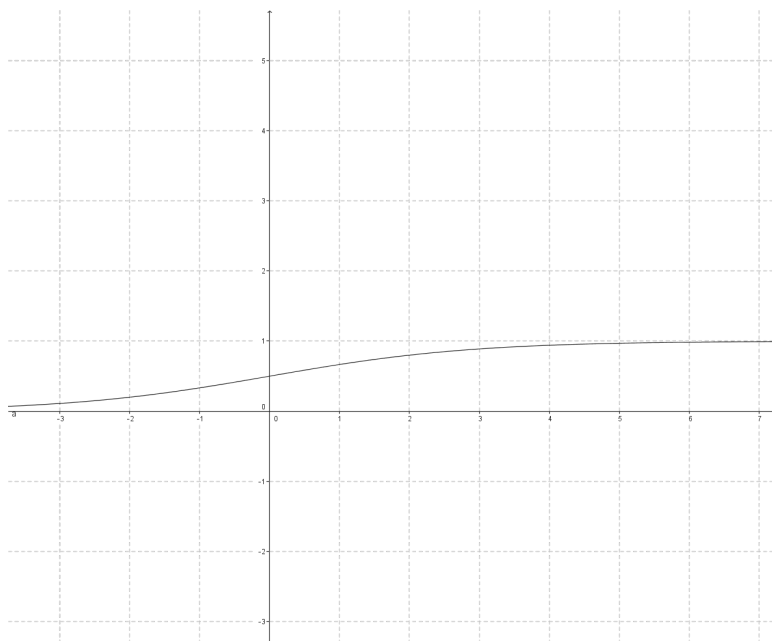
2. For each part in Problem 1, state the domain, D , and range, R , of the inverse function.

3. Sketch the graph of the inverse function for each of the following functions:

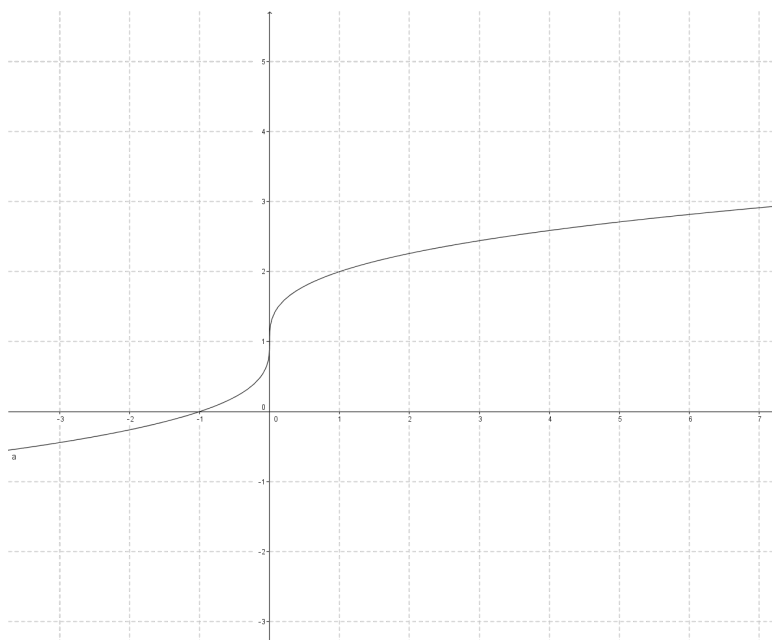
a.



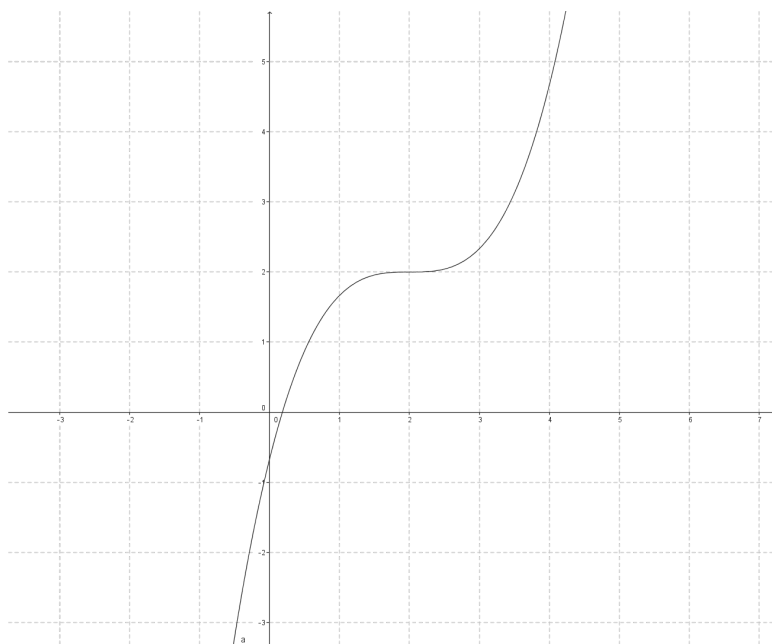
b.



c.

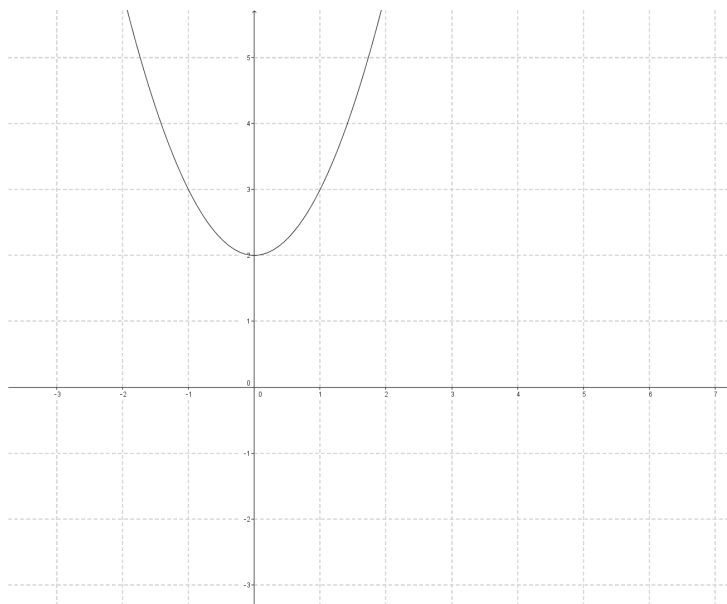


d.

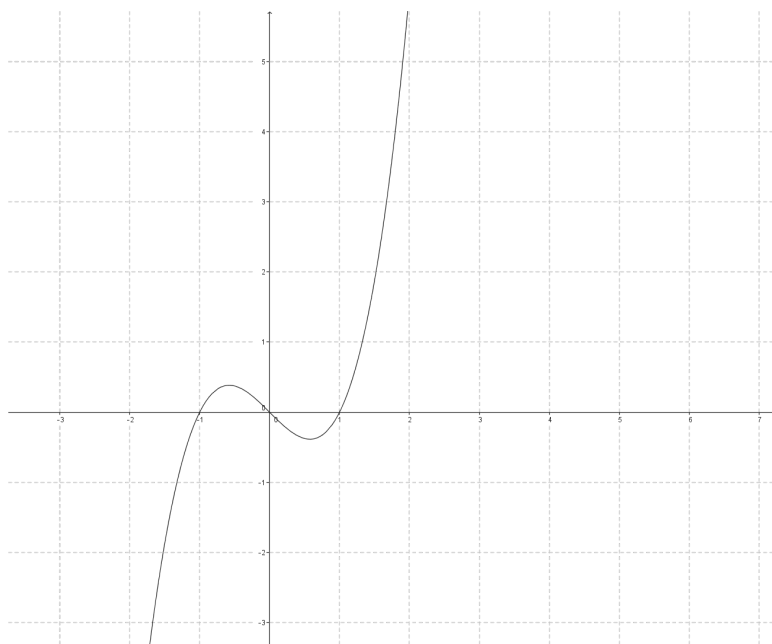


4. Natalie thinks that the inverse of $f(x) = x - 5$ is $g(x) = 5 - x$. To justify her answer, she calculates $f(5) = 0$ and then finds $g(0) = 5$, which gives back the original input.
- What is wrong with Natalie's reasoning?
 - Show that Natalie is incorrect by using other examples from the domain and range of f .
 - Find $f^{-1}(x)$. Where do f^{-1} and g intersect?
5. Sketch a graph of the inverse of each function graphed below by reflecting the graph about the line $y = x$. State whether or not the inverse is a function.

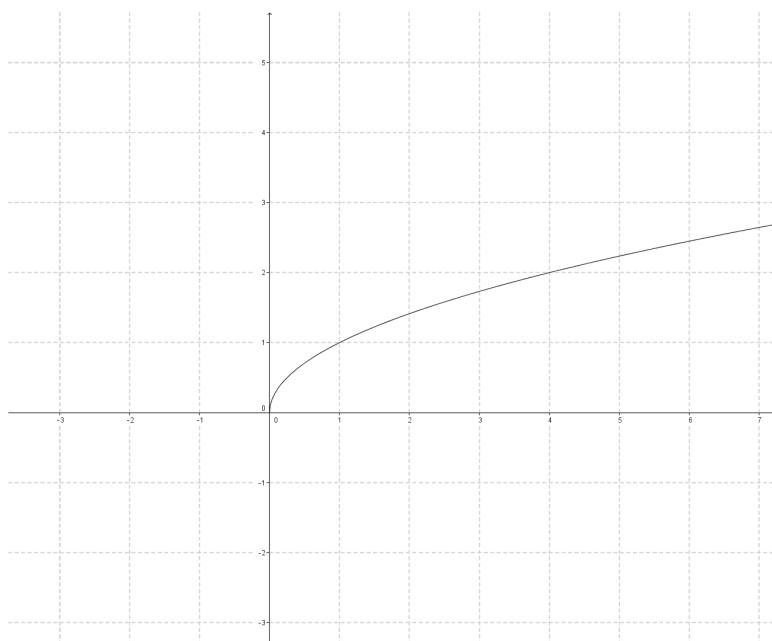
a.



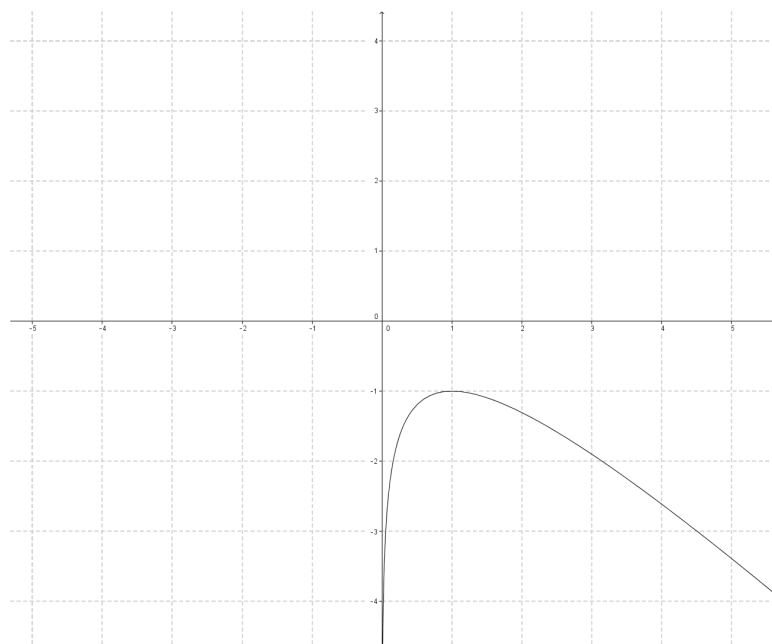
b.



c.



d.



6. How can you tell before you reflect a graph over $y = x$ if its reflection will be a function or not?
7. After finding several inverses, Callahan exclaims that every invertible linear function intersects its inverse at some point. What needs to be true about the linear functions that Callahan is working with for this to be true? What is true about linear functions that do not intersect their inverses?
8. If f is an invertible function such that $f(x) > x$ for all x , then what do we know about the inverse of f ?
9. Gavin purchases a new \$2,995 computer for his business, and when he does his taxes for the year, he is given the following information for deductions on his computer (this method is called MACRS—Modified Accelerated Cost Recovery System):

Period	Calculation for Deduction	Present Value
First Year	$D_1 = P_0 / 5 \times 200\% \times 50\%$	$P_0 - D_1 = P_1$
Second Year	$D_2 = P_1 / 5 \times 200\%$	$P_1 - D_2 = P_2$
Third Year	$D_3 = P_2 / 5 \times 200\%$	$P_2 - D_3 = P_3$

Where P_0 represents the value of the computer new.

- a. Construct a table for the function D , giving the deduction Gavin can claim in year x for his computer, $x = \{1, 2, 3\}$.
- b. Find the inverse of D .
- c. Construct a table for the function P , giving the present value of Gavin's computer in year x , $x = \{0, 1, 2, 3\}$.
- d. Find the inverse of P .

10. Problem 9 used the MACRS method to determine the possible deductions Gavin could have for the computer he purchased. The straight-line method can be used also. Assume the computer has a salvage value of \$500 after 5 years of use; call this value S . Then, Gavin would be presented with this information when he does his taxes:

Period	Calculation for Deduction	Present Value
First Year	$D_1 = (P_0 - S)/5 \times 50\%$	$P_0 - D_1 = P_1$
Second Year	$D_2 = (P_0 - S)/5$	$P_1 - D_2 = P_2$
Third Year	$D_3 = (P_0 - S)/5$	$P_2 - D_3 = P_3$
Fourth Year	$D_4 = (P_0 - S)/5$	$P_3 - D_4 = P_4$
Fifth Year	$D_5 = (P_0 - S)/5$	S

- Construct a table for the function D , giving the deduction Gavin can claim in year x for his computer in $x = \{1, 2, 3, 4, 5\}$.
- What do you notice about the function for deduction in this problem compared to the function in Problem 9?
- If you are given the deduction that Gavin claims in a particular year using the straight-line method, is it possible for you to know what year he claimed it in? Explain. What does this tell us about the inverse of D ?

Extension:

11. For each function in Problem 1, verify that the functions are inverses by composing the function with the inverse you found (in each case, after applying both functions, you should end up with the original input).

Lesson 19: Restricting the Domain

Classwork

Opening Exercise

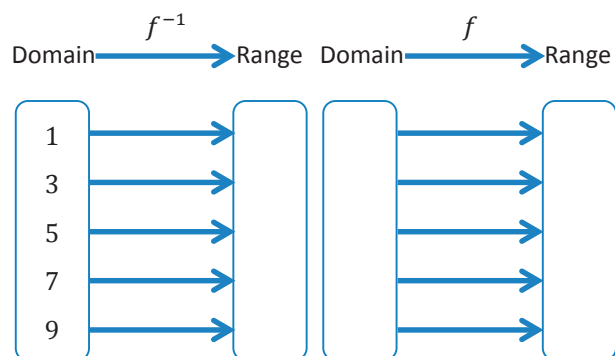
The function f with domain $\{1, 2, 3, 4, 5\}$ is shown in the table below.

x	$f(x)$
1	7
2	3
3	1
4	9
5	5

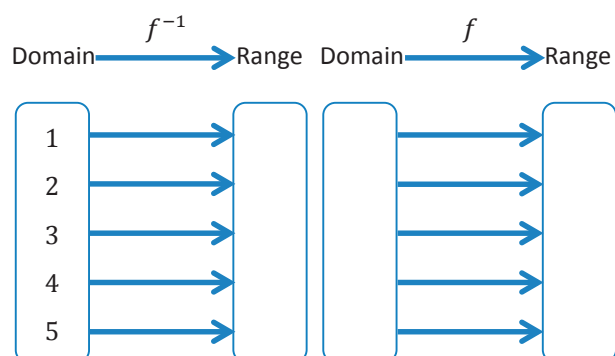
- What is $f(1)$? Explain how you know.
- What is $f^{-1}(1)$? Explain how you know.
- What is the domain of f^{-1} ? Explain how you know.
- Construct a table for the function f^{-1} , the inverse of f .

Exercises

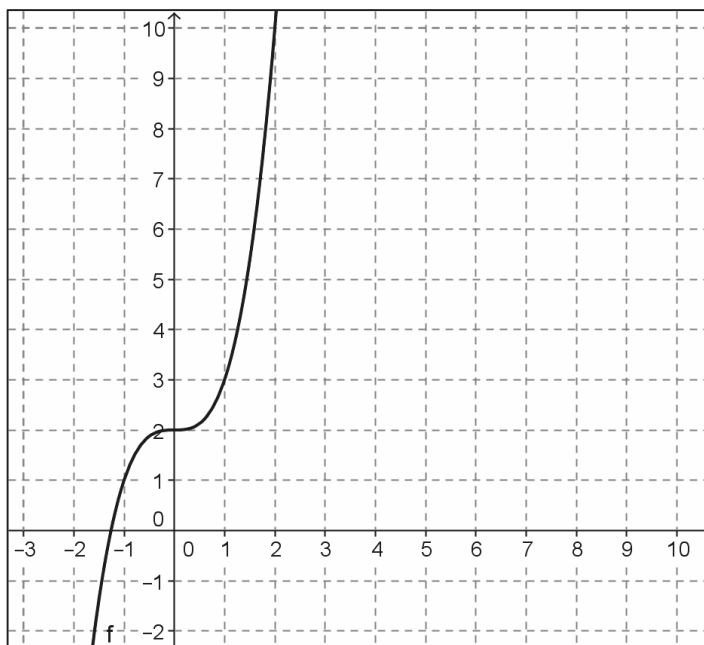
1. Complete the mapping diagram to show that $f(f^{-1}(x)) = x$.



2. Complete the mapping diagram to show that $f^{-1}(f(x)) = x$.



3. The graph of f is shown below.



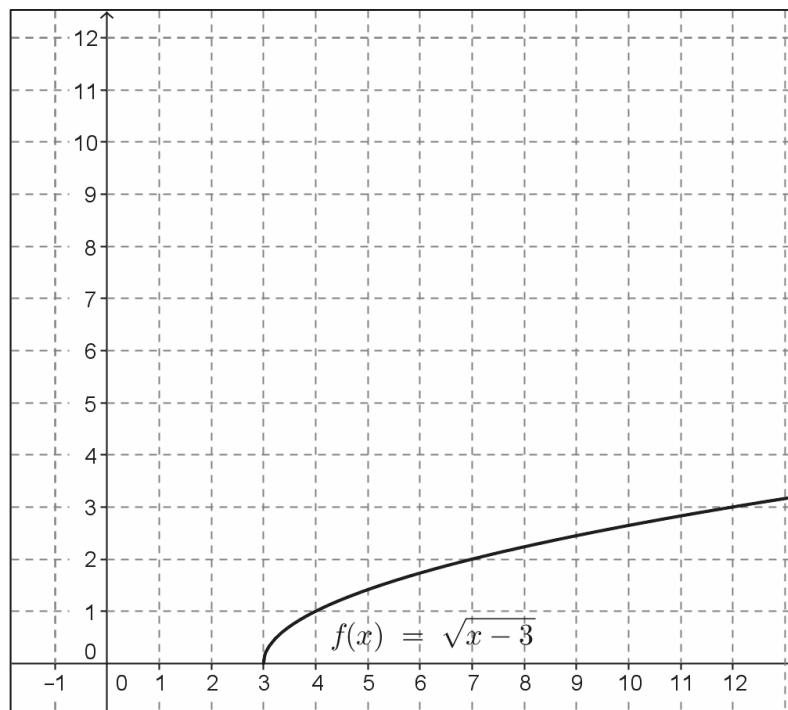
- a. Select several ordered pairs on the graph of f , and use those to construct a graph of f^{-1} in part (b).

- b. Draw the line $y = x$, and use it to construct the graph of f^{-1} below.



- c. The algebraic function for f is given by $f(x) = x^3 + 2$. Is the formula for $f^{-1}(x) = \sqrt[3]{x} - 2$? Explain why or why not.

4. The graph of $f(x) = \sqrt{x-3}$ is shown below. Construct the graph of f^{-1} .

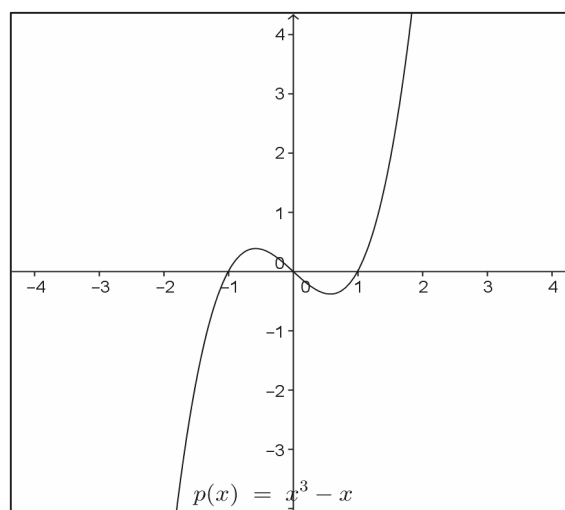
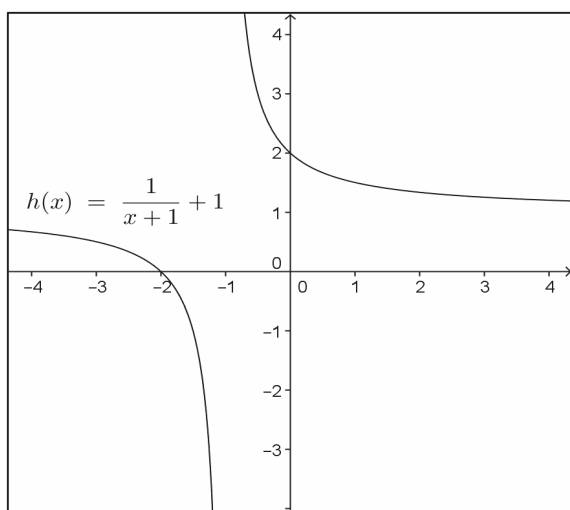
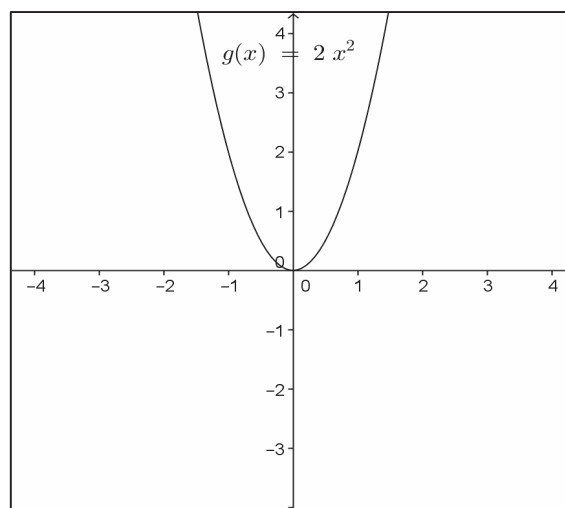
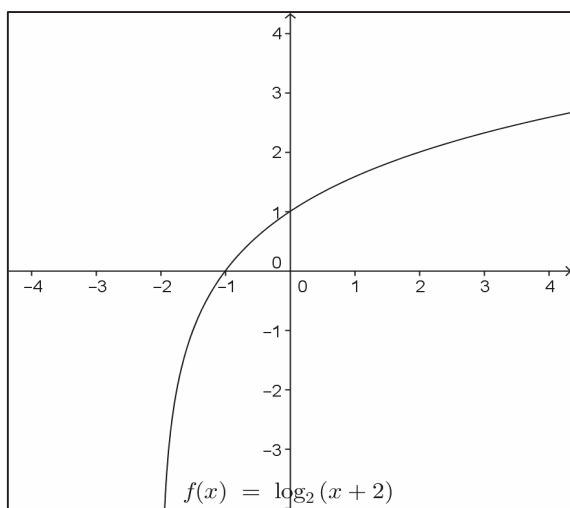


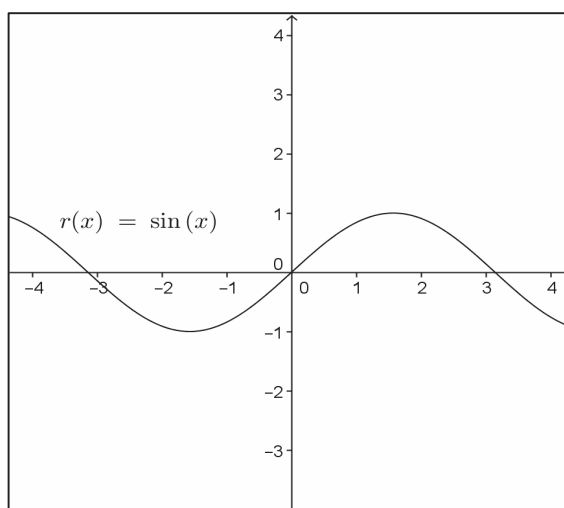
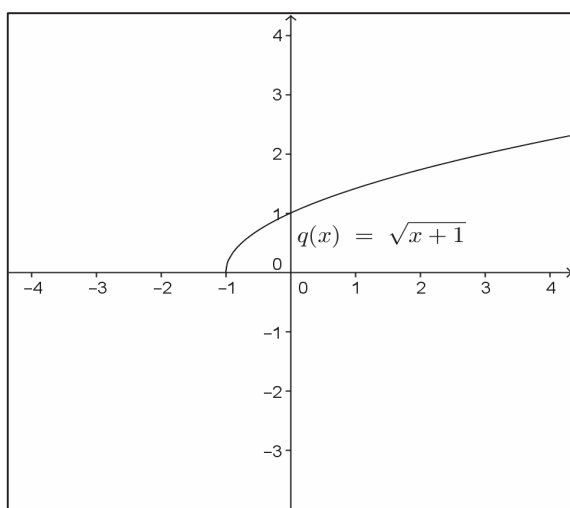
5. Morgan used the procedures learned in Lesson 18 to define $f^{-1}(x) = x^2 + 3$. How does the graph of this function compare to the one you made in Exercise 5?

6. Construct the inverse of the function f given by the table below. Is the inverse a function? Explain your reasoning.

x	-3	-2	-1	0	1	2	3
$f(x)$	4	-1	-4	-5	-4	-1	4

7. The graphs of several functions are shown below. Which ones are invertible? Explain your reasoning.





8. Given the function $f(x) = x^2 - 4$:

a. Select a suitable domain for f that makes it an invertible function. State the range of f .

b. Write a formula for f^{-1} . State the domain and range of f^{-1} .

c. Verify graphically that f , with the domain you selected, and f^{-1} are indeed inverses.

d. Verify that f and f^{-1} are indeed inverses by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

9. Three pairs of functions are given below. For which pairs are f and g inverses of each other? Show work to support your reasoning. If a domain is not specified, assume it is the set of real numbers.

a. $f(x) = \frac{x}{x+1}$, $x \neq -1$ and $g(x) = \frac{-x}{x-1}$, $x \neq 1$

b. $f(x) = \sqrt{x} - 1$, $x \geq 0$ and $g(x) = (x + 1)^2$

c. $f(x) = -0.75x + 1$ and $g(x) = -\frac{4}{3}x - \frac{4}{3}$

Lesson Summary

COMPOSITION OF A FUNCTION AND ITS INVERSE: To verify that two functions are inverses, show that $f(g(x)) = x$ and $g(f(x)) = x$.

INVERTIBLE FUNCTION: The domain of a function f can be restricted to make it invertible. A function is said to be invertible if its inverse is also a function.

Problem Set

- Let f be the function that assigns to each student in the class her biological mother.
 - In order for f to have an inverse, what condition must be true about students in the class?
 - If we enlarged the domain to include all students in the school, would this larger domain function have an inverse? Explain.
- Consider a linear function of the form $f(x) = mx + b$, where m and b are real numbers, and $m \neq 0$.
 - Explain why linear functions of this form always have an inverse that is also a function.
 - State the general form of a line that does not have an inverse.
 - What kind of function is the inverse of an invertible linear function (e.g., linear, quadratic, exponential, logarithmic, rational)?
 - Find the inverse of a linear function of the form $f(x) = mx + b$, where m and b are real numbers, and $m \neq 0$.
- Consider a quadratic function of the form $f(x) = b\left(\frac{x-h}{a}\right)^2 + k$ for real numbers a , b , h , k , and a , $b \neq 0$.
 - Explain why quadratic functions never have an inverse without restricting the domain.
 - What are the coordinates of the vertex of the graph of f ?
 - State the possible domains you can restrict f on so that it has an inverse.
 - What kind of function is the inverse of a quadratic function on an appropriate domain?
 - Find f^{-1} for each of the domains you gave in part (c).
- Show that $f(x) = mx + b$ for real numbers m and b with $m \neq 0$ has an inverse that is also a function.
- Explain why $f(x) = a(x - h)^2 + k$ for real numbers a , h , and k with $a \neq 0$ does not have an inverse that is a function. Support your answer in at least two different ways (numerically, algebraically, or graphically).

Extension:

6. Consider the function $f(x) = \sin(x)$.
- Graph $y = f(x)$ on the domain $[-2\pi, 2\pi]$.
 - If we require a restricted domain on f to be continuous and cover the entirety of the range of f , how many possible choices for a domain are there in your graph from part (a)? What are they?
 - Make a decision on which restricted domain you listed in part (b) makes the most sense to choose. Explain your decision.
 - Use a calculator to evaluate $\sin^{-1}(0.75)$ to three decimal places. How can you use your answer to find other values ψ such that $\sin(\psi) = 1$? Verify that your technique works by checking it against your graph in part (a).

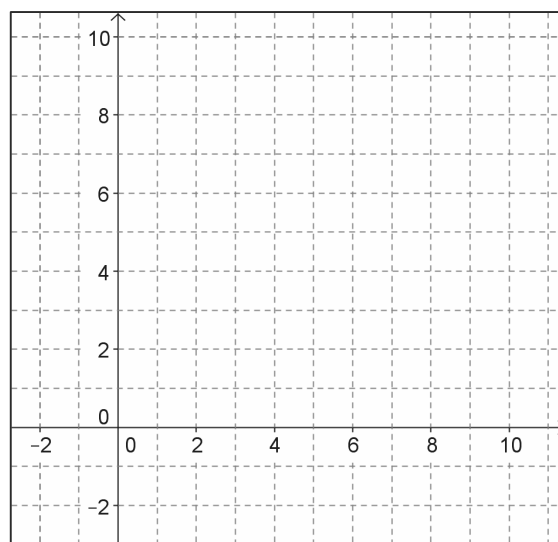
Lesson 20: Inverses of Logarithmic and Exponential Functions

Classwork

Opening Exercise

Let $f(x) = 2^x$.

- a. Complete the table, and use the points $(x, f(x))$ to create a sketch of the graph of $y = f(x)$.



x	$f(x)$
-2	
-1	
0	
1	
2	
3	

- b. Create a table of values for the function f^{-1} , and sketch the graph of $y = f^{-1}(x)$ on the grid above.

- c. What type of function is f^{-1} ? Explain how you know.

Example

Given $f(x) = 2^x$, use the definition of the inverse of a function and the definition of a logarithm to write a formula for $f^{-1}(x)$.

Exercises

1. Find the value of y in each equation. Explain how you determined the value of y .

a. $y = \log_2(2^2)$

b. $y = \log_2(2^5)$

c. $y = \log_2(2^{-1})$

d. $y = \log_2(2^x)$

2. Let $f(x) = \log_2(x)$ and $g(x) = 2^x$.

a. What is $f(g(x))$?

b. Based on the results of part (a), what can you conclude about the functions f and g ?

3. Find the value of y in each equation. Explain how you determined the value of y .

a. $y = 3^{\log_3(3)}$

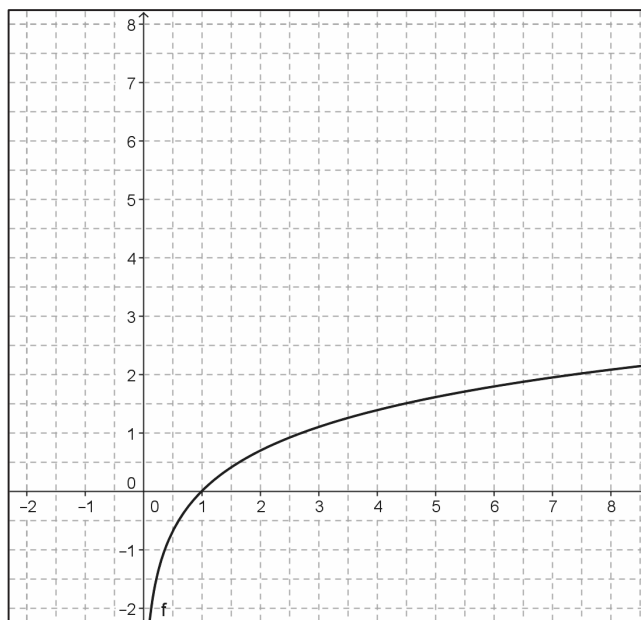
b. $y = 3^{\log_3(9)}$

c. $y = 3^{\log_3(81)}$

d. $y = 3^{\log_3(x)}$

4. Let $f(x) = \log_3(x)$ and $g(x) = 3^x$.
- What is $g(f(x))$?
 - Based on the results in part (a), what can you conclude about the functions f and g ?
5. Verify by composition that the functions $f(x) = b^x$ and $g(x) = \log_b(x)$ for $b > 0$ are inverses of one another.

6. The graph of $y = f(x)$, a logarithmic function, is shown below.



- a. Construct the graph of $y = f^{-1}(x)$.
- b. Estimate the base b of these functions. Explain how you got your answer.
7. Use a calculator to get a very accurate estimate of the irrational number e .
8. Is the graph of $y = f^{-1}(x)$ in Exercise 6 a good approximation of the function $g(x) = e^x$? Explain your reasoning.

9. Show that $f(x) = \ln(x)$ and $g(x) = e^x$ are inverse functions by graphing $y = f(g(x))$ and $y = g(f(x))$ on a graphing calculator. Explain how your graphs support the fact that these two functions are indeed inverses of one another.
10. What is the base of the natural logarithm function $f(x) = \ln(x)$? Explain how you know.
11. Find the inverse of each function.
- a. $f(x) = 2^{x-3}$
- b. $g(x) = 2 \log(x - 1)$
- c. $h(x) = \ln(x) - \ln(x - 1)$

d. $k(x) = 5 - 3^{-\frac{x}{2}}$

12. Check your solutions to Exercise 11 by graphing the functions and the inverses that you found and verifying visually that the reflection property holds.

Problem Set

1. Find the inverse of each function.

a. $f(x) = 3^x$

b. $f(x) = \left(\frac{1}{2}\right)^x$

c. $g(x) = \ln(x - 7)$

d. $h(x) = \frac{\log_3(x+2)}{\log_3(5)}$

e. $f(x) = 3(1.8)^{0.2x} + 3$

f. $g(x) = \log_2(\sqrt[3]{x-4})$

g. $h(x) = \frac{5^x}{5^x + 1}$

h. $f(x) = 2^{-x+1}$

i. $g(x) = \sqrt{\ln(3x)}$

j. $h(x) = e^{\frac{1}{5}x+3} - 4$

2. Consider the composite function $f \circ g$, composed of invertible functions f and g .

- Either $f^{-1} \circ g^{-1}$ or $g^{-1} \circ f^{-1}$ is the inverse of the composite function. Which one is it? Explain.
- Show via composition of functions that your choice of $(f \circ g)^{-1}$ was the correct choice. (Hint: Function composition is associative.)

3. Let $m(x) = \frac{x}{x-1}$.

- Find the inverse of m .
- Graph m . How does the graph of m explain why this function is its own inverse?
- Think of another function that is its own inverse.

Extension:

4. One of the definitions of e involves the infinite series $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!} + \cdots$. A generalization exists to define e^x :

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots$$

This series definition of e^x allows us to approximate powers of the transcendental number e using strictly rational numbers. This definition is accurate for all real numbers.

- Verify that the formula given for e can be obtained by plugging $x = 1$ into the formula for e^x .
- Use the first seven terms of the series to calculate e , e^2 , and e^3 .
- Use the inverse of $y = e^x$ to see how accurate your answer to part (b) is.
- Newer calculators and computers use these types of series carried out to as many terms as needed to produce their results for operations that are not otherwise obvious. It may seem cumbersome to calculate these by hand knowing that computers can calculate hundreds and thousands of terms of these series in a single second. Use a calculator or computer to compare how accurate your results from part (b) were to the value given by your technology.
- $\ln\left(\frac{x}{x-1}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + \cdots + \frac{1}{nx^n} + \cdots$ for $|x| > 1$. What does your response to Exercise 3 part (a) tell you that $\ln(x)$ is equal to?

Lesson 21: Logarithmic and Exponential Problem Solving

Classwork

Woolly mammoths, elephant-like mammals, have been extinct for thousands of years. In the last decade, several well-preserved woolly mammoths have been discovered in the permafrost and icy regions of Siberia. Using a technique called *radiocarbon (Carbon-14) dating*, scientists have determined that some of these mammoths died nearly 40,000 years ago.

This technique was introduced in 1949 by the American chemist Willard Libby and is one of the most important tools archaeologists use for dating artifacts that are less than 50,000 years old. Carbon-14 is a radioactive isotope present in all organic matter. Carbon-14 is absorbed in small amounts by all living things. The ratio of the amount of normal carbon (Carbon-12) to the amount of Carbon-14 in all living organisms remains nearly constant until the organism dies. Then, the Carbon-14 begins to decay because it is radioactive.

Exploratory Challenge/Exercises 1–14

By examining the amount of Carbon-14 that remains in an organism after death, one can determine its age. The half-life of Carbon-14 is 5,730 years, meaning that the amount of Carbon-14 present is reduced by a factor of $\frac{1}{2}$ every 5,730 years.

1. Complete the table.

Years Since Death	0	5,730							
C-14 Atoms Remaining Per 1.0×10^8 C-12 Atoms	10,000								

Let C be the function that represents the number of C-14 atoms remaining per 1.0×10^8 C-12 atoms t years after death.

2. What is $C(11,460)$? What does it mean in this situation?

3. Estimate the number of C-14 atoms per 1.0×10^8 C-12 atoms you would expect to remain in an organism that died 10,000 years ago.
4. What is $C^{-1}(625)$? What does it represent in this situation?
5. Suppose the ratio of C-14 to C-12 atoms in a recently discovered woolly mammoth was found to be 0.000 001. Estimate how long ago this animal died.
6. Explain why the $C^{-1}(100)$ represents the answer to Exercise 5.
7. What type of function best models the data in the table you created in Exercise 1? Explain your reasoning.
8. Write a formula for C in terms of t . Explain the meaning of any parameters in your formula.

9. Graph the set of points $(t, C(t))$ from the table and the function C to verify that your formula is correct.
10. Graph the set of points $(C(t), t)$ from the table. Draw a smooth curve connecting those points. What type of function would best model this data? Explain your reasoning.

11. Write a formula that gives the years since death as a function of the amount of C-14 remaining per 1.0×10^8 C-12 atoms.
12. Use the formulas you have created to accurately calculate the following:
- The amount of C-14 atoms per 1.0×10^8 C-12 atoms remaining in a sample after 10,000 years
 - The years since death of a sample that contains 100 C-14 atoms per 1.0×10^8 C-12 atoms

c. $C(25,000)$

d. $C^{-1}(1,000)$

13. A baby woolly mammoth that was discovered in 2007 died approximately 39,000 years ago. How many C-14 atoms per 1.0×10^8 C-12 atoms would have been present in the tissues of this animal when it was discovered?
14. A recently discovered woolly mammoth sample was found to have a red liquid believed to be blood inside when it was cut out of the ice. Suppose the amount of C-14 in a sample of the creature's blood contained 3,000 atoms of C-14 per 1.0×10^8 atoms of C-12. How old was this woolly mammoth?

Exercises 15–18

Scientists can infer the age of fossils that are older than 50,000 years by using similar dating techniques with other radioactive isotopes. Scientists use radioactive isotopes with half-lives even longer than Carbon-14 to date the surrounding rock in which the fossil is embedded.

A general formula for the amount A of a radioactive isotope that remains after t years is

$$A = A_0 \left(\frac{1}{2} \right)^{\frac{t}{h}}$$

where A_0 is the amount of radioactive substance present initially and h is the half-life of the radioactive substance.

15. Solve this equation for t to find a formula that infers the age of a fossil by dating the age of the surrounding rocks.

16. Let $(x) = A_0 \left(\frac{1}{2} \right)^{\frac{x}{h}}$. What is $A^{-1}(x)$?

17. Verify that A and A^{-1} are inverses by showing that $A(A^{-1}(x)) = x$ and $A^{-1}(A(x)) = x$.

18. Explain why, when determining the age of organic materials, archaeologists and anthropologists would prefer to use the logarithmic function to relate the amount of a radioactive isotope present in a sample and the time since its death.

Problem Set

1. A particular bank offers 6% interest per year compounded monthly. Timothy wishes to deposit \$1,000.
 - a. What is the interest rate per month?
 - b. Write a formula for the amount A Timothy has after n months.
 - c. Write a formula for the number of months it takes Timothy to have A dollars.
 - d. Doubling-time is the amount of time it takes for an investment to double. What is the doubling-time of Timothy's investment?
 - e. In general, what is the doubling-time of an investment with an interest rate of $\frac{r}{12}$ per month?
2. A study done from 1950 through 2000 estimated that the world population increased on average by 1.77% each year. In 1950, the world population was 2,519 million.
 - a. Write a formula for the world population t years after 1950. Use p to represent the world population.
 - b. Write a formula for the number of years it takes to reach a population of p .
 - c. Use your equation in part (b) to find when the model predicts that the world population is 10 billion.
3. Consider the case of a bank offering r (given as a decimal) interest per year compounded monthly, if you deposit $\$P$.
 - a. What is the interest rate per month?
 - b. Write a formula for the amount A you have after n months.
 - c. Write a formula for the number of months it takes to have A dollars.
 - d. What is the doubling-time of an investment earning 7% interest per year, compounded monthly? Round up to the next month.
4. A half-life is the amount of time it takes for a radioactive substance to decay by half. In general, we can use the equation $A = P \left(\frac{1}{2}\right)^t$ for the amount of the substance remaining after t half-lives.
 - a. What does P represent in this context?
 - b. If a half-life is 20 hours, rewrite the equation to give the amount after h hours.
 - c. Use the natural logarithm to express the original equation as having base e .
 - d. The formula you wrote in part (c) is frequently referred to as the "Pert" formula, that is, Pe^{rt} . Analyze the value you have in place for r in part (c). What do you notice? In general, what do you think r represents?
 - e. Jess claims that any exponential function can be written with base e ; is she correct? Explain why.
5. If caffeine reduces by about 10% per hour, how many hours h does it take for the amount of caffeine in a body to reduce by half (round up to the next hour)?

6. Iodine-123 has a half-life of about 13 hours, emits gamma-radiation, and is readily absorbed by the thyroid. Because of these facts, it is regularly used in nuclear imaging.
- Write a formula that gives you the percent p of iodine-123 left after t half-lives.
 - What is the decay rate per hour of iodine-123? Approximate to the nearest millionth.
 - Use your result to part (b). How many hours h would it take for you to have less than 1% of an initial dose of iodine-123 in your system? Round your answer to the nearest tenth of an hour.
7. An object heated to a temperature of 50°C is placed in a room with a constant temperature of 10°C to cool down. The object's temperature T after t minutes can be given by the function $T(t) = 10 + 40e^{-0.023105t}$.
- How long does it take for the object to cool down to 30°C ?
 - Does it take longer for the object to cool from 50°C to 30°C or from 30°C to 10.1°C ?
 - Will the object ever be 10°C if kept in this room?
 - What is the domain of T^{-1} ? What does this represent?
8. The percent of usage of the word *judgment* in books can be modeled with an exponential decay curve. Let P be the percent as a function of x , and let x be the number of years after 1900; then, $P(x) = 0.0220465 \cdot e^{-0.0079941x}$.
- According to the model, in what year was the usage 0.1% of books?
 - When does the usage of the word *judgment* drop below 0.001% of books? This model was made with data from 1950 to 2005. Do you believe your answer is accurate? Explain.
 - Find P^{-1} . What does the domain represent? What does the range represent?