A Story of Functions®

Eureka Math™

Precalculus, Module 3

Teacher Edition

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Rational and Exponential Functions

[1]Each lesson is ONE day, and ONE day is considered a 45-minute period.

Precalculus and Advanced Topics • Module 3

Rational and Exponential Functions

OVERVIEW

Students encountered the fundamental theorem of algebra, that every polynomial function has at least one zero in the realm of the complex numbers (**N-CN.C.9**), in Algebra II Module 1. Topic A of this module brings students back to the study of complex roots of polynomial functions. Students first briefly review quadratic and cubic functions and then extend familiar polynomial identities to both complex numbers (**N-CN.C.8**) and to general polynomial functions. Students use polynomial identities to find square roots of complex numbers. The binomial theorem and its relationship to Pascal's triangle are explored using roots of unity (**A-APR.C.5**). Topic A concludes with students' use of Cavalieri's principle to derive formulas (**G-GPE.A.3**) for the volume of the sphere and other geometric solids (**G-GMD.A.2**).

In Topic B, students explore composition of functions in depth (**F-BF.A.1c**) and notice that a composition of a polynomial function with the function $f(x) = \frac{1}{x}$ gives functions that can be written as ratios of polynomial functions. A study of rational expressions shows that these expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression (**A-APR.D.7**). Focus then turns to restricted domain values as students compare the properties of rational functions represented in different ways (**F-IF.C.9**). Students apply these operations to simplify rational expressions and go on to graphing rational functions, identifying zeros and asymptotes, and analyzing end behavior (**F-IF.C.7d**).

The module ends with Topic C, in which students study inverse functions, being careful to understand when inverse functions do and do not exist, working to restrict the domain of a function to produce an invertible function. They compare and create different representations of functions including tables and graphs. Students compose functions to verify that one function is the inverse of another and work with tables of data to identify features of inverse functions (**F-BF.B.4b**, **F-BF.B.4c**, **F-BF.B.4d**). Special emphasis is given to the inverse relationship between exponential and logarithmic functions (**F-BF.B.5**).

Focus Standards

Use complex numbers in polynomial identities and equations.

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.

N-CN.C.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Use polynomial identities to solve problems.

A-APR.C.5 (+) Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[2]

Rewrite rational expressions.

A-APR.D.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Analyze functions using different representations.

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior

F-IF.C.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*[3]

Build a function that models a relationship between two quantities.

F-BF.A.1 Write a function that describes a relationship between two quantities.★

c. (+) Compose functions. *For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.*

Build new functions from existing functions.

F-BF.B.4 Find inverse functions.

b. (+) Verify by composition that one function is the inverse of another.

c. (+) Read value of an inverse function from a graph or a table, given that the function has an inverse.

d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

F-BF.B.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

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[2]The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.
[3]This standard is to be applied to rational functions.

Explain volume formulas and use them to solve problems.

G-GMD.A.2 (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

Translate between the geometric description and the equation for a conic section.

G-GPE.A.3 (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Foundational Standards

Reason quantitatively and use units to solve problems.

N-Q.A.2 Define appropriate quantities for the purpose of descriptive modeling.★

Perform arithmetic operations with complex numbers.

N-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

N-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.

Interpret the structure of expressions.

A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of* P *and a factor not depending on* P.

Write expressions in equivalent forms to solve problems.

A-SSE.B.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $\left(1.15^{1/12}\right)^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is* 15%*.*

Create equations that describe numbers or relationships.★

A-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

A-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance* R*.*

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Solve systems of equations.

A-REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Experiment with transformations in the plane.

G-CO.A.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Extend the domain of trigonometric functions using the unit circle.

F-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$, and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

Prove and apply trigonometric identities.

F-TF.C.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Focus Standards for Mathematical Practice

MP.3 **Construct viable arguments and critique the reasoning of others.** Students construct arguments and critique the reasoning of others when making conjectures about the roots of polynomials (a polynomial of degree 3 will have three roots) and solving problems by applying algebraic properties. Students determine the domain and range of rational functions and reason what effect these restrictions have on the graph of the rational function. Students use reasoning to argue that restricting the domain of a function allows for the construction of an inverse function.

MP.7 **Look for and make use of structure.** Students use polynomial identities to determine roots of polynomials and square roots of complex numbers. They relate the structure of rational expressions to the graphs of rational functions by studying transformations of these graphs. Students determine the relationship between functions and their inverses.

MP.8 **Look for and express regularity in repeated reasoning.** Students use prior knowledge of the fundamental theorem of algebra to justify the number of roots of unity. Students develop understanding of the binomial theorem through repeated binomial expansions and connecting their observations to patterns in Pascal's triangle. In performing and reasoning about several computations with fractional expressions, students extend the properties of rational numbers to rational expressions.

Terminology

New or Recently Introduced Terms

- **Ellipse** (An *ellipse* is the set of all points in a plane such that the sum of the distances from two points (foci) to any point on the line is constant. Given k, foci A and B, and any point P on the ellipse, $PA + PB = k$.)

  Standard equation for an ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

  An ellipse whose foci points are the same point, that is, $A = B$, is a circle.
- **Horizontal Asymptote** (Let L be a real number. The line given by the equation $y = L$ is a *horizontal asymptote* of the graph of $y = f(x)$ if at least one of the following statements is true.
  - As x approaches infinity, $f(x)$ approaches L.
  - As x approaches negative infinity, $f(x)$ approaches L.)
- **Hyperbola** (A *hyperbola* is the set of points in a plane whose distances to two fixed points A and B, called the foci, have a constant difference. Given P and a positive constant, k, $|PA - PB| = k$.)

  Standard equation for a hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
- **Vertical Asymptote** (Let a be a real number. The line given by the equation $x = a$ is a *vertical asymptote* of the graph of $y = f(x)$ if at least one of the following statements is true.
  - As x approaches a, $f(x)$ approaches infinity.
  - As x approaches a, $f(x)$ approaches negative infinity.)

Familiar Terms and Symbols[4]

- Complex Numbers
- Domain
- Exponential Function
- Inverse Functions
- Logarithmic Function
- Polar Form
- Sphere
- Transformation
- Volume

[4]These are terms and symbols students have seen previously.

Suggested Tools and Representations

- Graphing Calculator
- Wolfram Alpha Software
- Geometer's Sketchpad Software
- Geogebra Software

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic A	Constructed response with rubric	N-CN.C.8, N-CN.C.9, A-APR.C.5, G-GMD.A.2, G-GPE.A.3
End-of-Module Assessment Task	After Topic C	Constructed response with rubric	A-APR.C.7, F-IF.C.7d, F-IF.C.9, F-BF.A.1c, F-BF.B.4b, F-BF.B.4c, F-BF.B.4d, F-BF.B.5

Topic A

Polynomial Functions and the Fundamental Theorem of Algebra

N-CN.C.8, N-CN.C.9, A-APR.C.5, G-GMD.A.2, G-GPE.A.3

Focus Standards:	N-CN.C.8	(+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.
	N-CN.C.9	(+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.
	A-APR.C.5	(+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.
	G-GMD.A.2	(+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.
	G-GPE.A.3	(+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.
Instructional Days:	9	
Lesson 1:	Solutions to Polynomial Equations (P)[1]	
Lesson 2:	Does Every Complex Number Have a Square Root? (P)	
Lesson 3:	Roots of Unity (E)	
Lessons 4–5:	The Binomial Theorem (P, P)	
Lesson 6:	Curves in the Complex Plane (P)	
Lessons 7–8:	Curves from Geometry (S, S)	
Lesson 9:	Volume and Cavalieri's Principle (S)	

In Algebra II Module 1, students encountered the fundamental theorem of algebra, which states that every polynomial function has at least one zero in the realm of the complex numbers (**N-CN.C.9**). Topic A of this module brings students back to the study of both real and complex roots of polynomial functions. Students first complete a brief review of quadratic and cubic functions by extending familiar polynomial identities to the complex numbers (**N-CN.C.8**) and then review general polynomial functions.

[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

In Lesson 1, students work with equations with complex number solutions and apply identities such as $a^2 + b^2 = (a + bi)(a - bi)$ for real numbers a and b to solve equations and explore the implications of the fundamental theorem of algebra (**N-CN.C.8** and **N-CN.C.9**). In Precalculus and Advanced Topics Module 1, students used the polar form of a complex number to find powers and roots of complex numbers. However, nearly all of the examples used in that module involved complex numbers with arguments that were multiples of special angles ($\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$). In Lesson 2, we return to the rectangular form of a complex number to show algebraically that we can find the square roots of any complex number without having to express it first in polar form. Students use the properties of complex numbers and the fundamental theorem of algebra to find the square roots of any complex number by creating and solving polynomial equations (**N-CN.C.8** and **N-CN.C.9**). Lesson 3 connects the work within the algebra strand of solving polynomial equations to the geometry and arithmetic of complex numbers in the complex plane. Students determine solutions to polynomial equations using various methods and interpret the results in the real and complex planes. Students extend factoring to the complex numbers (**N-CN.C.8**) and more fully understand the conclusion of the fundamental theorem of algebra (**N-CN.C.9**) by seeing that a polynomial of degree n has n complex roots when they consider the roots of unity graphed in the complex plane. These lessons can be supported with graphing software that is available free of charge such as GeoGebra.

In Lesson 4, students carry out the tedious process of repeatedly multiplying binomial factors together, leading them to see the usefulness of finding a quicker way to expand binomials raised to whole number powers. By exploring binomial expansions, students recognize patterns in the coefficients of terms illustrated in Pascal's triangle. They verify patterns in the powers of terms in binomial expansions and are introduced to the formal binomial theorem, which they prove by combinatorics. Students then apply the binomial theorem to expand binomials and to find specific terms in expansions (**A-APR.C.5**). Lesson 5 provides students with opportunities to explore additional patterns formed by the coefficients of binomial expansions. They apply the binomial theorem to find a mathematical basis for the patterns observed. They also apply the theorem to explore the average rate of change for the volume of a sphere with a changing radius (**A-APR.C.5**).

In Lesson 6, students review the characteristics of the graphs of the numbers $z = r(\cos(\theta) + i\sin(\theta))$, recognizing that they represent circles centered at the origin with radius equal to the modulus. They explore sets of complex numbers written in the form $z = a\cos(\theta) + bi\sin(\theta)$, identifying the graphs as ellipses. Students convert between the complex and real forms of equations for ellipses, including those whose centers are not the origin (**G-GPE.A.3**). In Lesson 7, students study ellipses as the set of all points in a plane such that the sum of the distances from two points (foci) to any point on the line containing the foci is constant. Formally, an ellipse is a set of points that satisfy a constraint $PF + PG = k$, where points F and G are the foci of the ellipse, P is any point distinct from points F and G, and k is a constant. The goal of this lesson is to connect these two definitions. That is, if a point $P(x, y)$ satisfies the constraint $PF + PG = k$, then it also satisfies an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for suitable values of a and b (**G-GPE.A.3**).

Lesson 8 introduces students to the hyperbola as they graph hyperbolas and derive the standard equation of a hyperbola knowing its foci (**G-GPE.A.3**). Students learn that the hyperbola is a set of points in a plane whose distances to two fixed points (foci) have a constant difference. Students see the connection between ellipses, parabolas, and hyperbolas in the context of the orbital path of a satellite. Topic A ends with Lesson 9 as students see ellipses and hyperbolas as cross sections of a cone. Cavalieri's volume principle is based on cross-sectional area and is used to explore the volume of pyramids, cones, and spheres (**G-GMD.A.2**). Students were first introduced to this principle in Geometry (Module 3 Lesson 10) when studying the volume of prisms and cylinders. In this lesson, students use the principle to find the volumes of pyramids, cones, and spheres.

Throughout Topic A, students make conjectures about the roots of polynomials and solve problems by applying algebraic properties. They construct arguments and critique the reasoning of others in these algebraic contexts (MP.3). They examine the structure of expressions to support their solutions and make generalizations about roots of polynomials and to find patterns of the coefficients of expanded polynomials (MP.7 and MP.8).

Lesson 1: Solutions to Polynomial Equations

Student Outcomes

- Students determine all solutions of polynomial equations over the set of complex numbers and understand the implications of the fundamental theorem of algebra, especially for quadratic polynomials.

Lesson Notes

Students studied polynomial equations and the nature of the solutions of these equations extensively in Algebra II Module 1, extending factoring to the complex realm. The fundamental theorem of algebra indicates that any polynomial function of degree n will have n zeros (including repeated zeros). Establishing the fundamental theorem of algebra was one of the greatest achievements of nineteenth-century mathematics. It is worth spending time further exploring it now that students have a much broader understanding of complex numbers. This lesson reviews what they learned in previous grades and provides additional support for their understanding of what it means to solve polynomial equations over the set of complex numbers. Students work with equations with complex number solutions and apply identities such as $a^2 + b^2 = (a + bi)(a - bi)$ for real numbers a and b to solve equations and explore the implications of the fundamental theorem of algebra (**N-CN.C.8** and **N-CN.C.9**). Throughout the lesson, students vary their reasoning by applying algebraic properties (MP.3) and examining the structure of expressions to support their solutions and make generalizations (MP.7 and MP.8). Relevant definitions introduced in Algebra II are provided in the student materials for this lesson.

A note on terminology: Equations have solutions, and functions have zeros. The distinction is subtle but important. For example, the equation $(x - 1)(x - 3) = 0$ has solutions 1 and 3, while the polynomial function $p(x) = (x - 1)(x - 3)$ has zeros at 1 and 3. Zeros of a function are the x-intercepts of the graph of the function; they are also known as *roots*.

Classwork

Opening Exercise (3 minutes)

Use this Opening Exercise to activate prior knowledge about polynomial equations. Students may need to be reminded of the definition of a polynomial from previous grades. Have students work this exercise independently and then quickly share their answers with a partner. Lead a short discussion using the questions below.

Scaffolding:

- Remind students of the definition of a polynomial equation by using a Frayer model. For an example, see Module 1 Lesson 5.
- Be sure to include examples ($2x + 3 = 0$, $x^2 - 4 = 0$, $(x - 3)(x + 2) = 2x - 5$, and $x^5 = 1$) and non-examples ($\sin(2x) - 1 = 0$, $\frac{1}{x} = 5$, and $2^{3x} - 1 = 5$).

Opening Exercise

How many solutions are there to the equation $x^2 = 1$? Explain how you know.

There are two solutions to the equation: 1 and -1. I know these are the solutions because they make the equation true when each value is substituted for x.

- How do you know that there aren't any more real number solutions?
  - *If you sketch the graph of $f(x) = x^2$ and the graph of the line $y = 1$, they intersect in exactly two points. The x-coordinates of the intersection points are the solutions to the equation.*
- How can you show algebraically that this equation has just two solutions?
  - *Rewrite the equation $x^2 = 1$, and solve it by factoring. Then, apply the zero product property.*

$$x^2 - 1 = 0 \textit{ or } (x - 1)(x + 1) = 0$$

$$x - 1 = 0 \textit{ or } x + 1 = 0$$

$$x = 1 \textit{ or } x = -1$$

  - *So, the solutions are 1 and -1.*
- You just found and justified why this equation has only two real number solutions. How do we know that there aren't any complex number solutions to $x^2 = 1$?
  - *We would have to show that a second-degree polynomial equation has exactly two solutions over the set of complex numbers. (Another acceptable answer would be the fundamental theorem of algebra, which states that a second-degree polynomial equation has at most two solutions. Since we have found two solutions that are real, we know we have found all possible solutions.)*
- Why do the graphical approach and the algebraic approach not clearly provide an answer to the previous question?
  - *The graphical approach assumes you are working with real numbers. The algebraic approach does not clearly eliminate the possibility of complex number solutions. It only shows two real number solutions.*

Example 1 (5 minutes): Prove that a Quadratic Equation Has Only Two Solutions over the Set of Complex Numbers

This example illustrates an approach to showing that 1 and -1 are the only real or complex solutions to the quadratic equation $x^2 = 1$. Students may not have seen this approach before. However, they should be very familiar with operations with complex numbers after their work in Algebra II Module 1 and Module 1 of this course. We work with solutions to $x^n = 1$ in later lessons, and this approach is most helpful.

MP.3

Example 1: Prove that a Quadratic Equation Has Only Two Solutions over the Set of Complex Numbers

Prove that 1 and -1 are the only solutions to the equation $x^2 = 1$.

Let $x = a + bi$ be a complex number so that $x^2 = 1$.

a. **Substitute $a + bi$ for x in the equation $x^2 = 1$.**

$(a + bi)^2 = 1$

b. **Rewrite both sides in standard form for a complex number.**

$a^2 + 2abi + b^2i^2 = 1$
$(a^2 - b^2) + 2abi = 1 + 0i$

Scaffolding:
Ask advanced learners to develop the proof in Example 1 on their own without the leading questions.

MP.3

c. Equate the real parts on each side of the equation and equate the imaginary parts on each side of the equation.

$a^2 - b^2 = 1$ and $2ab = 0$

d. Solve for a and b, and find the solutions for $x = a + bi$.

For $2ab = 0$, either $a = 0$ or $b = 0$.

If $a = 0$, then $-b^2 = 1$, which gives us $b = i$ or $b = -i$.

Substituting into $x = a + bi$ gives us $0 + i \cdot i = -1$ or $0 - i \cdot i = 1$.

If $b = 0$, then $a^2 = 1$, which gives us $a = 1$ or $a = -1$.

Substituting into $x = a + bi$ gives us $1 + 0 \cdot i = 1$ or $-1 + 0 \cdot i = -1$.

Thus, the only complex number solutions to this equation are the complex numbers $1 + 0i$ or $-1 + 0i$.

The quadratic formula also proves that the only solutions to the equation are 1 and -1 even if we solve the equation over the set of complex numbers. If this did not come up earlier in this lesson as students shared their thinking on the Opening Exercise, discuss it now.

- What is the quadratic formula?
  - *The formula that provides the solutions to a quadratic equation when it is written in the form $ax^2 + bx + c = 0$ where $a \neq 0$.*
  - *If $ax^2 + bx + c = 0$ and $a \neq 0$, then $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$, and the solutions to the equation are the complex numbers $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$.*
- How does the quadratic formula guarantee that a quadratic equation has at most two solutions over the set of complex numbers?
  - *The quadratic formula is a general solution to the equation $ax^2 + bx + c = 0$, where $a \neq 0$. The formula shows two solutions: $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$. There is only one solution if $b^2 - 4ac = 0$. There are two distinct real number solutions when $b^2 - 4ac > 0$, and there are two distinct complex number solutions when $b^2 - 4ac < 0$.*

Exercises 1–6 (5 minutes)

Allow students time to work these exercises individually, and then discuss as a class. Note that this is very similar to the Opening Exercise used in Algebra II Module 1 Lesson 40, with an added degree of difficulty since the coefficients of the polynomial are also complex. Exercises 1–6 are a review of patterns in the factors of the polynomials below for real numbers a and b.

$$a^2 - b^2 = (a + b)(a - b)$$

$$a^2 + b^2 = (a + bi)(a - bi)$$

Exercises 1 and 2 ask students to multiply binomials that have a product that is a sum or difference of squares. Exercises 3–6 ask students to factor polynomials that are sums and differences of squares.

Exercises

Find the product.

1. $(z-2)(z+2)$

   $z^2 - 4$

2. $(z+3i)(z-3i)$

   $z^2 + 9$

Write each of the following quadratic expressions as the product of two linear factors.

3. $z^2 - 4$

   $(z+2)(z-2)$

4. $z^2 + 4$

   $(z+2i)(z-2i)$

5. $z^2 - 4i$

   $(z+2\sqrt{i})(z-2\sqrt{i})$ *or* $\left(z+(1+i)\sqrt{2}\right)\left(z-(1+i)\sqrt{2}\right)$ [*using the fact that* $\sqrt{i} = \frac{1+i}{\sqrt{2}}$]

6. $z^2 + 4i$

   $(z+2i\sqrt{i})(z-2i\sqrt{i})$ *or* $\left(z+(i-1)\sqrt{2}\right)\left(z-(i-1)\sqrt{2}\right)$ [*using the fact that* $\sqrt{i} = \frac{1+i}{\sqrt{2}}$]

Students may be curious about the square roots of a complex number. They may recall from Precalculus Module 1, that we studied these when considering the polar form of a complex number. This question is also addressed in Lesson 2 from an algebraic perspective.

- How did we know that each quadratic expression could be factored into two linear terms?
  - *The fundamental theorem of algebra guarantees that a polynomial of degree* 2 *can be factored into* 2 *linear factors. We proved this was true for quadratic expressions by using the solutions produced with the quadratic formula to write the expression as two linear factors.*
- Does the fundamental theorem of algebra apply even if the coefficients are non-real numbers?
  - *It still held true for Exercises 3 and 4, so it seems to, at least if the constant is a non-real number.*

Exercises 7–10 (10 minutes)

Students should work the following exercises in small groups. Have different groups come to the board and present their solutions to the class.

7. **Can a quadratic polynomial equation with real coefficients have one real solution and one complex solution? If so, give an example of such an equation. If not, explain why not.**

The quadratic formula shows that if the discriminant $b^2 - 4ac$ is negative, both solutions are complex numbers that are complex conjugates. If it is positive, both solutions are real. If it is zero, there is one (repeated) real solution. We cannot have a real solution coupled with a complex solution because complex solutions occur in conjugate pairs.

Scaffolding:

Provide several factored quadratic polynomial equations, and ask students to identify the solutions and write them in standard form.

$$(x-2)(x+2)=0$$
$$(2x-3)(2x+3)=0$$
$$(x-2i)(x+2i)=0$$
$$(2x-i)(2x+i)=0$$
$$(x-1+i)(x-1-i)=0$$
$$\big(x-(1+2i)\big)\big(x-(1-2i)\big)=0$$
$$\big(x-1+\sqrt{3}\big)\big(x-1-\sqrt{3}\big)=0$$

Recall from Algebra II that every quadratic expression can be written as a product of two linear factors, that is,

$$ax^2+bx+c=a(x-r_1)(x-r_2),$$

where r_1 and r_2 are solutions of the polynomial equation $ax^2+bx+c=0$.

8. **Solve each equation by factoring, and state the solutions.**

   a. $x^2+25=0$

   $(x+5i)(x-5i)=0$

   The solutions are $5i$ and $-5i$.

   b. $x^2+10x+25=0$

   $(x+5)(x+5)=0$

   The solution is -5.

9. **Give an example of a quadratic equation with $2+3i$ as one of its solutions.**

We know that if $2+3i$ is a solution of the equation, then its conjugate $2-3i$ must also be a solution.

$$\big(x-(2+3i)\big)\big(x-(2-3i)\big)=\big((x-2)-3i\big)\big((x-2)+3i\big)$$

Using the structure of this expression, we have $(a-bi)(a+bi)$ where $a=x-2$ and $b=3$.
Since $(a+bi)(a-bi)=a^2+b^2$ for all real numbers a and b,

$$\begin{aligned}(x-2-3i)(x-2+3i)&=(x-2)^2+3^3\\&=x^2-4x+4+9\\&=x^2-4x+13.\end{aligned}$$

10. **A quadratic polynomial equation with real coefficients has a complex solution of the form $a+bi$ with $b\neq 0$. What must its other solution be and why?**

The other solution is $a-bi$. If the polynomial equation must have real coefficients, then $\big(x-(a+bi)\big)\big(x-(a-bi)\big)$ when multiplied must yield an expression with real number coefficients.

$$\begin{aligned}\big(x-(a+bi)\big)\big(x-(a-bi)\big)&=(x-a-bi)(x-a+bi)\\&=(x-a)^2-b^2i^2\\&=(x-a)^2+b^2\\&=x^2-2ax+a^2+b^2\end{aligned}$$

Since a and b are real numbers, this expression always has real number coefficients.

Debrief these exercises by having different groups share their approaches. Give students enough time to struggle with these exercises in their small groups. Use the results of these exercises to further plan for reteaching if students cannot recall what they learned in Algebra I and Algebra II.

Discussion (5 minutes)

Use this discussion to help students recall the fundamental theorem of algebra first introduced in Algebra II Module 1 Lesson 40.

- What are the solutions to the polynomial equation $(x-1)(x+2i)(x-2i)=0$? What is the degree of this equation?
  - *The solutions are* 1, $2i$, *and* $-2i$. *This is a third-degree equation.*
- What are the solutions to the polynomial equation $(x-1)(x+1)(x-2i)(x+2i)=0$? What is the degree of this equation?
  - *The solutions are* 1, -1, $2i$, *and* $-2i$. *This is a fourth-degree equation.*
- Predict how many solutions the equation $x^5-3x^3+2x=0$ has. Justify your response.
  - *It should have at most* 5 *solutions. It is a fifth-degree polynomial.*

MP.7 & MP.8

- To find the solutions, we need to write x^5-3x^3+2x as a product of 5 linear factors. Explain how to factor this polynomial.
  - *Factor out* x, *giving us* $x(x^4-3x^2+2)$.

> *Scaffolding:*
>
> Have advanced learners factor x^4-3x^2+2 without the hint.

- How can you factor (x^4-3x^2+2)?
  - *We know* $x^4=(x^2)^2$, *so let* $u=x^2$. *This gives us a polynomial* u^2-3u+2 *in terms of* u, *which factors into* $(u-1)(u-2)$. *Now substitute* x^2 *for* u, *and we have* $(x^2-1)(x^2-2)$.
- What is the factored form of the equation?
  - $x(x^2-1)(x^2-2)=x(x+1)(x-1)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)=0$
- What are the solutions to the equation $x^5-3x^3+2x=0$?
  - *The solutions are* 0, 1, -1, $\sqrt{2}$, *and* $-\sqrt{2}$.
- How many solutions does a degree n polynomial equation have? Explain your reasoning.
  - *If every polynomial equation can be written as the product of* n *linear factors, then there are at most* n *solutions.*

This is an appropriate point to reintroduce the fundamental theorem of algebra. For more details or to provide additional background information, refer back to Algebra II Module 1.

> Fundamental Theorem of Algebra
>
> 1. Every polynomial function of degree $n \geq 1$ with real or complex coefficients has at least one real or complex zero.
> 2. Every polynomial of degree $n \geq 1$ with real or complex coefficients can be factored into n linear terms with real or complex coefficients.

Continue the discussion, and have students write any examples and their summaries in the space below each question. Have students discuss both of these questions in their small groups before leading a whole-group discussion.

- Could a polynomial function of degree n have more than n zeros? Explain your reasoning.
  - *If a polynomial function has n zeros, then it has n linear factors, which means the degree is n.*
- Could a polynomial function of degree n have less than n zeros? Explain your reasoning.
  - *Yes. If a polynomial has repeated linear factors, then it has less than n distinct zeros. For example, $p(x) = (x-1)^n$ has only one zero: the number 1.*

Exercises 11–15 (10 minutes)

Give students time to work on the exercises either individually or with a partner, and then share answers as a class. On Exercises 11 and 12, students need to recall polynomial division from Algebra II. Students divided polynomials using both the reverse tabular method and long division. Consider reviewing one or both of these methods with students. In Exercise 11, students may recall the sum and difference of cube formulas derived through polynomial long division in Algebra II. Students should have access to technology to aid with problems such as Exercise 12. However, in Exercise 12, consider asking students to verify that $x = 2$ is a zero of p rather than having them use technology to locate the zero.

11. Write the left side of each equation as a product of linear factors, and state the solutions.

a. $x^3 - 1 = 0$

$$0 = x^3 - 1 = (x-1)(x^2 + x + 1)$$

Using the quadratic formula, $x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$, so the quadratic expression $x^2 + x + 1$ factors into

$$x^2 + x + 1 = \left(x - \left(\frac{-1 + i\sqrt{3}}{2}\right)\right)\left(x - \left(\frac{-1 - i\sqrt{3}}{2}\right)\right).$$

Then, the factored form of the equation is

$$(x-1)\left(x - \left(\frac{-1 + i\sqrt{3}}{2}\right)\right)\left(x - \left(\frac{-1 - i\sqrt{3}}{2}\right)\right) = 0.$$

The solutions are 1, $\frac{-1+i\sqrt{3}}{2}$, and $\frac{-1-i\sqrt{3}}{2}$.

b. $x^3 + 8 = 0$

$$x^3 + 8 = (x+2)(x^2 - 2x + 4)$$

Using the quadratic formula, $x = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{-3}}{2} = 1 \pm i\sqrt{3}$, so the quadratic expression $x^2 - 2x + 4$ factors into

$$x^2 \pm 2 + 4 = \left(x - \left(1 + i\sqrt{3}\right)\right)\left(x - \left(1 - i\sqrt{3}\right)\right).$$

Then, the factored form of the equation is

$$(x+2)\left(x - \left(1 + i\sqrt{3}\right)\right)\left(x - \left(1 - i\sqrt{3}\right)\right) = 0.$$

The solutions are -2, $1 + i\sqrt{3}$, and $1 - i\sqrt{3}$.

Scaffolding:

Display the sum and difference of cube formulas on the board once students have completed Exercise 11 as a reminder for future work.

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

c. $x^4 + 7x^2 + 10 = 0$

$$(x^2+5)(x^2+2) = (x+i\sqrt{5})(x-i\sqrt{5})(x+i\sqrt{2})(x-i\sqrt{2}) = 0$$

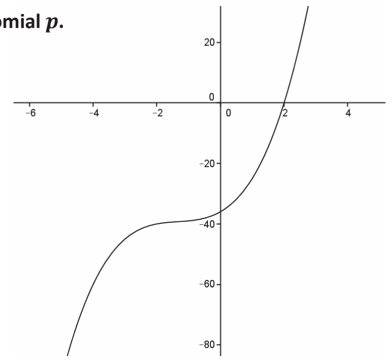
The solutions are $\pm i\sqrt{5}$ and $\pm i\sqrt{2}$.

12. Consider the polynomial $p(x) = x^3 + 4x^2 + 6x - 36$.

MP.5

a. Graph $y = x^3 + 4x^2 + 6x - 36$, and find the real zero of polynomial p.

The graph of p has an x-intercept at $x = 2$. Therefore, $x = 2$ is a zero of p.



b. Write p as a product of linear factors.

$p(x) = (x-2)(x+3-3i)(x+3+3i)$

c. What are the solutions to the equation $p(x) = 0$?

The solutions are 2, $-3+3i$, and $-3-3i$.

13. Malaya was told that the volume of a box that is a cube is $4,096$ cubic inches. She knows the formula for the volume of a cube with side length x is $V(x) = x^3$, so she models the volume of the box with the equation $x^3 - 4096 = 0$.

a. Solve this equation for x.

$x^3 - 4096 = (x-16)(x^2+16x+256) = 0$

$x = 16, x = -8+8\sqrt{3}i, x = -8-8\sqrt{3}i$

b. Malaya shows her work to Tiffany and tells her that she has found three different values for the side length of the box. Tiffany looks over Malaya's work and sees that it is correct but explains to her that there is only one valid answer. Help Tiffany explain which answer is valid and why.

Since we are looking for the dimensions of a box, only real solutions are acceptable, so the answer is 16 inches.

14. Consider the polynomial $p(x) = x^6 - 2x^5 + 7x^4 - 10x^3 + 14x^2 - 8x + 8$.

a. Graph $y = x^6 - 2x^5 + 7x^4 - 10x^3 + 14x^2 - 8x + 8$, and state the number of real zeros of p.

There are no real zeros.

b. Verify that i is a zero of p.

$$\begin{aligned} p(i) &= i^6 - 2i^5 + 7i^4 - 10i^3 + 14i^2 - 8i + 8 \\ &= -1 - 2i + 7 + 10i - 14 - 8i + 8 \\ &= 0 \end{aligned}$$

c. Given that i is a zero of p, state another zero of p.

Another zero is $-i$.

MP.3

d. Given that $2i$ and $1+i$ are also zeros of p, explain why polynomial p cannot possibly have any real zeros.

Since $2i$ and $1+i$ are zeros, $-2i$ and $1-i$ must also be zeros of p. The fundamental theorem of algebra tells us that since p is a degree-6 polynomial it can be written as a product of 6 linear factors. We now know that p has 6 complex zeros and therefore cannot have any real zeros.

e. **What is the solution set to the equation $p(x) = 0$?**

The solution set is $\{i, -i, 2i, -2i, 1+i, 1-i\}$.

15. **Think of an example of a sixth-degree polynomial equation that, when written in standard form, has integer coefficients, four real number solutions, and two imaginary number solutions. How can you be sure your equation will have integer coefficients?**

One correct response is $(x-1)(x+1)(x-2)(x+2)(x-2i)(x+2i) = 0$. By selecting an imaginary number bi, where b is an integer, and its conjugate as solutions, we know by the identity $(a+bi)(a-bi) = a^2 + b^2$ that these factors produce a quadratic expression with integer coefficients. If we choose the real solutions to also be integers, then, when written in standard form, the polynomial equation has integer coefficients.

Closing (4 minutes)

Review the information in the Lesson Summary box by asking students to choose one vocabulary term, theorem, or identity and paraphrase it with a partner. Select students to share their paraphrasing with the class.

Lesson Summary

Relevant Vocabulary

POLYNOMIAL FUNCTION: Given a polynomial expression in one variable, a *polynomial function in one variable* is a function $f: \mathbb{R} \to \mathbb{R}$ such that for each real number x in the domain, $f(x)$ is the value found by substituting the number x into all instances of the variable symbol in the polynomial expression and evaluating.

It can be shown that if a function $f: \mathbb{R} \to \mathbb{R}$ is a *polynomial function*, then there is some nonnegative integer n and collection of real numbers $a_0, a_1, a_2, \ldots, a_n$ with $a_n \neq 0$ such that the function satisfies the equation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

for every real number x in the domain, which is called the *standard form of the polynomial function.* The function $f(x) = 3x^3 + 4x^2 + 4x + 7$, where x can be any real number, is an example of a function written in standard form.

DEGREE OF A POLYNOMIAL FUNCTION: The *degree of a polynomial function* is the degree of the polynomial expression used to define the polynomial function. The degree is the highest degree of its terms.

The degree of $f(x) = 8x^3 + 4x^2 + 7x + 6$ is 3, but the degree of $g(x) = (x+1)^2 - (x-1)^2$ is 1 because when g is put into standard form, it is $g(x) = 4x$.

ZEROS OR ROOTS OF A FUNCTION: A *zero* (or *root*) *of a function* $f: \mathbb{R} \to \mathbb{R}$ is a number x of the domain such that $f(x) = 0$. A *zero of a function* is an element in the solution set of the equation $f(x) = 0$.

Given any two polynomial functions p and q, the solution set of the equation $p(x)q(x) = 0$ can be quickly found by solving the two equations $p(x) = 0$ and $q(x) = 0$ and combining the solutions into one set.

A number a is zero of a polynomial function p with multiplicity m if the factored form of p contains $(x-a)^m$.

Every polynomial function of degree n, for $n \geq 1$, has n zeros over the complex numbers, counted with multiplicity. Therefore, such polynomials can always be factored into n linear factors.

Exit Ticket (3 minutes)

Name ____ Date ____________________

Lesson 1: Solutions to Polynomial Equations

Exit Ticket

1. Find the solutions of the equation $x^4 - x^2 - 12$. Show your work.

2. The number 1 is a zero of the polynomial $p(x) = x^3 - 3x^2 + 7x - 5$.

   a. Write $p(x)$ as a product of linear factors.

   b. What are the solutions to the equation $x^3 - 3x^2 + 7x - 5 = 0$?

Exit Ticket Sample Solutions

1. Find the solutions of the equation $x^4 - x^2 - 12$. Show your work.

$$x^4 - x^2 - 12 = (x^2+3)(x^2-4) = (x - i\sqrt{3})(x + i\sqrt{3})(x-2)(x+2) = 0$$

The solutions of the equation are $i\sqrt{3}$, $-i\sqrt{3}$, 2, -2.

2. The number 1 is a zero of the polynomial $p(x) = x^3 - 3x^2 + 7x - 5$.

   a. Write $p(x)$ as a product of linear factors.

   $(x-1)(x^2-2x+5)$

   $(x-1)(x-(1+2i))(x-(1-2i))$

   b. What are the solutions to the equation $x^3 - 3x^2 + 7x - 5 = 0$?

   The solutions are 1, $1+2i$, *and* $1-2i$.

Problem Set Sample Solutions

1. Find all solutions to the following quadratic equations, and write each equation in factored form.

   a. $x^2 + 25 = 0$

   $x = \pm 5i$,
   $(x+5i)(x-5i) = 0$

   b. $-x^2 - 16 = -7$

   $x = \pm 3i$,
   $(x+3i)(x-3i) = 0$

   c. $(x+2)^2 + 1 = 0$

   $x^2 + 4x + 5 = 0, \qquad x = \dfrac{-4 \pm \sqrt{16-20}}{2} = \dfrac{-4 \pm 2i}{2} = -2 \pm i$

   $(x+2+i)(x+2-i) = 0$

   d. $(x+2)^2 = x$

   $x^2 + 3x + 4 = 0, \qquad x = \dfrac{-3 \pm \sqrt{9-16}}{2} = \dfrac{-3 \pm \sqrt{3}i}{2} = -\dfrac{3}{2} \pm \dfrac{\sqrt{3}i}{2}$

   $\left(x + \dfrac{3}{2} + \dfrac{\sqrt{3}i}{2}\right)\left(x + \dfrac{3}{2} - \dfrac{\sqrt{3}i}{2}\right) = 0$

   e. $(x^2+1)^2 + 2(x^2+1) - 8 = 0$

   $(x^2+1+4)(x^2+1-2) = 0, \qquad x^2 + 5 = 0, \qquad x = \pm\sqrt{5}i, \qquad x^2 - 1 = 0, \qquad x = \pm 1$

   $(x + \sqrt{5}i)(x - \sqrt{5}i)(x+1)(x-1) = 0$

f. $(2x-1)^2 = (x+1)^2 - 3$

$4x^2 - 4x + 1 - x^2 - 2x - 1 + 3 = 0, \quad 3x^2 - 6x + 3 = 0, \quad x^2 - 2x + 1 = 0,$

$(x-1)^2 = 0, \quad x = \pm 1$

$(x+1)(x-1) = 0$

g. $x^3 + x^2 - 2x = 0$

$x(x^2 + x - 2) = 0, \quad x(x+2)(x-1) = 0,$

$x = 0, \quad x = -2, \quad x = 1$

$x(x+2)(x-1) = 0$

h. $x^3 - 2x^2 + 4x - 8 = 0$

$x^2(x-2) + 4(x-2) = 0, \quad (x-2)(x^2+4) = 0, \quad x = 2, \quad x = \pm 2i$

$(x-2)(x+2i)(x-2i) = 0$

2. **The following cubic equations all have at least one real solution. Find the remaining solutions.**

a. $x^3 - 2x^2 - 5x + 6 = 0$

One real solution is -2*; then* $x^3 - 2x^2 - 5x + 6 = (x+2)(x^2 - 4x + 3) = (x+2)(x-1)(x-3)$.

The solutions are -2, 1, 3.

b. $x^3 - 4x^2 + 6x - 4 = 0$

One real solution is 2*; then* $x^3 - 4x^2 + 6x - 4 = (x-2)(x^2 - 2x + 2)$. *Using the quadratic formula on* $x^2 - 2x + 2 = 0$ *gives* $x = 1 \pm i$.

The solutions are 2, $1+i$, $1-i$.

c. $x^3 + x^2 + 9x + 9 = 0$

One real solution is -1*; then* $x^3 + x^2 + 9x + 9 = (x+1)(x^2+9) = (x+1)(x-3i)(x+3i)$.

The solutions are -1, $3i$, $-3i$.

d. $x^3 + 4x = 0$

One real solution is 0*; then* $(x^3 + 4x) = x(x^2+4) = x(x+2i)(x-2i)$.

The solutions are 0, $2i$, $-2i$.

e. $x^3 + x^2 + 2x + 2 = 0$

One real solution is $x = -1$*; then* $x^3 + x^2 + 2x + 2 = (x+1)(x^2+2) = (x+1)\left(x - i\sqrt{2}\right)\left(x + i\sqrt{2}\right)$.

The solutions are -1, $i\sqrt{2}$, $-i\sqrt{2}$.

3. **Find the solutions of the following equations.**

a. $4x^4 - x^2 - 18 = 0$

Set $u = x^2$. Then

$$\begin{aligned} 4x^4 - x^2 - 18 &= 4u^2 - u - 18 \\ &= (4u-9)(u+2) \\ &= (4x^2-9)(x^2+2) \\ &= (2x-3)(2x+3)\left(x+i\sqrt{2}\right)\left(x-i\sqrt{2}\right) \end{aligned}$$

The solutions are $i\sqrt{2}, -i\sqrt{2}, -\frac{3}{2}, \frac{3}{2}$.

b. $x^3 - 8 = 0$

$$\begin{aligned} (x^3-8) &= (x-2)(x^2+2x+4) \\ &= (x-2)\left(x-\left(-1-i\sqrt{3}\right)\right)\left(x-\left(-1+i\sqrt{3}\right)\right) \end{aligned}$$

The solutions are $2, -1+i\sqrt{3}, -1-i\sqrt{3}$.

c. $8x^3 - 27 = 0$

$$\begin{aligned} (8x^3-27) &= (2x-3)(4x^2+6x+9) \\ &= (2x-3)\left(2x-\left(\frac{-3-3i\sqrt{3}}{2}\right)\right)\left(2x-\left(\frac{-3+3i\sqrt{3}}{2}\right)\right) \end{aligned}$$

The solutions are $\frac{3}{2}, -\frac{3}{4}+\frac{3i\sqrt{3}}{4}, -\frac{3}{4}-\frac{3i\sqrt{3}}{4}$.

d. $x^4 - 1 = 0$

$$\begin{aligned} (x^4-1) &= (x^2-1)(x^2+1) \\ &= (x-1)(x+1)(x+i)(x-i) \end{aligned}$$

The solutions are $1, -1, i, -i$.

e. $81x^4 - 64 = 0$

$$\begin{aligned} (81x^4-64) &= (9x^2-8)(9x^2+8) \\ &= \left(3x-2\sqrt{2}\right)\left(3x+2\sqrt{2}\right)\left(3x+2i\sqrt{2}\right)\left(3x-2i\sqrt{2}\right) \end{aligned}$$

The solutions are $\frac{2\sqrt{2}}{3}i, -\frac{2\sqrt{2}}{3}i, \frac{2\sqrt{2}}{3}, -\frac{2\sqrt{2}}{3}$.

f. $20x^4 + 121x^2 - 25 = 0$

$$\begin{aligned} (20x^4+121x^2-25) &= (4x^2+25)(5x^2-1) \\ &= (2x+5i)(2x-5i)\left(\sqrt{5}x-1\right)\left(\sqrt{5}x+1\right) \end{aligned}$$

The solutions are $\frac{5}{2}i, -\frac{5}{2}i, \frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}$.

g. $64x^3 + 27 = 0$

$$(64x^3 + 27) = (4x + 3)(16x^2 - 12x + 9)$$
$$= (4x + 3)\left(8x - 3\left(1 - i\sqrt{3}\right)\right)\left(8x - 3\left(1 + i\sqrt{3}\right)\right)$$

The solutions are $\frac{3}{8} + \frac{3i\sqrt{3}}{8}, -\frac{3}{4}, \frac{3}{8} - \frac{3i\sqrt{3}}{8}$.

h. $x^3 + 125 = 0$

$$(x^3 + 125) = (x + 5)(x^2 - 5x + 25)$$
$$= (x + 5)\left(x - \frac{5\left(1 - i\sqrt{3}\right)}{2}\right)\left(x - \frac{5\left(1 + i\sqrt{3}\right)}{2}\right)$$

The solutions are $-5, \frac{5}{2} + \frac{5i\sqrt{3}}{2}, \frac{5}{2} - \frac{5i\sqrt{3}}{2}$.

Lesson 2: Does Every Complex Number Have a Square Root?

Student Outcomes

- Students apply their understanding of polynomial identities that have been extended to the complex numbers to find the square roots of complex numbers.

Lesson Notes

In Precalculus Module 1, students used the polar form of a complex number to find powers and roots of complex numbers. However, nearly all of the examples used in those lessons involved complex numbers with arguments that were multiples of special angles ($\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$). In this lesson, we return to the rectangular form of a complex number to show algebraically that we can find the square roots of any complex number without having to express it first in polar form. Students use the properties of complex numbers and the fundamental theorem of algebra to find the square roots of any complex number by creating and solving polynomial equations (**N-NC.C.8** and **N-NC.C.9**). While solving these equations, students see that we arrive at polynomial identities with two factors that guarantee two roots to the equation. Throughout the lesson, students use algebraic properties to justify their reasoning (MP.3) and examine the structure of expressions to support their solutions and make generalizations (MP.7 and MP.8).

Classwork

Opening (5 minutes)

Organize the class into groups of 3–5 students. Start by displaying the question shown below. Give students time to consider an answer to this question on their own, and then allow them to discuss it in small groups. Have a representative from each group briefly summarize their small group discussions.

- Does every complex number have a square root? If yes, provide at least two examples. If no, explain why not.
  - *If we think about transformations, then squaring a complex number dilates a complex number by the modulus and rotates it by the argument. Thus, taking the square root of a complex number should divide the argument by 2 and have a modulus equal to the square root of the original modulus.*

    For example, the complex number i would have a square root with modulus equal to 1 and argument equal to $45°$, which is $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

Scaffolding:

- Create an anchor chart with the following formulas:

  For any complex number z,

  $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

  The n^{th} roots of $z = re^{i\theta}$ are given by

  $\sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right)$

  for integers k and n such that $n > 0$ and $0 \leq k < n$.
- Ask students to explain the formulas above for given values of n, r, and θ. For example, substitute $\frac{\pi}{2}$ for θ, 1 for r, and 2 for n into the formula for the n^{th} roots of z, and ask students what the expressions represent (the square root of i).

Student discussions and responses should reveal how much they recall from their work in Module 1 Lessons 18 and 19. In those lessons, students used the polar form of a complex number to find the n^{th} roots of a complex number in polar form. Based on student responses to this question, the teacher may need to briefly review the polar form of a complex number and the formulas developed in Lessons 18 and 19 for finding powers of a complex number z and roots of a complex number z.

Exercises 1–6 (10 minutes)

Students should work these exercises with their group. As students work, be sure to circulate around the classroom to monitor the groups' progress. Pause as needed for whole-group discussion and debriefing, especially if a majority of the groups are struggling to make progress.

Exercises 1–6

1. **Use the geometric effect of complex multiplication to describe how to calculate a square root of $z = 119 + 120i$.**

   The square root of this number would have a modulus equal to the square root of $|119 + 120i|$ and an argument equal to $\frac{\arg(119+120i)}{2}$.

2. **Calculate an estimate of a square root of $119 + 120i$.**

   $$|z| = \sqrt{119^2 + 120^2} = 169$$
   $$\arg(z) = \arctan\left(\frac{120}{119}\right)$$

   The square root's modulus is $\sqrt{169}$ or 13, and the square root's argument is $\frac{1}{2}\arctan\left(\frac{120}{119}\right)$. The square root is close to $13(\cos(22.6°) + i\sin(22.6°))$ or $12 + 4.99i$.

If students get stuck on the next exercises, lead a short discussion to help set the stage for establishing that every complex number has two square roots that are opposites of each other.

- What are the square roots of 4?
  - *The square roots of 4 are 2 and -2 because $(2)^2 = 4$ and $(-2)^2 = 4$.*
- What are the square roots of 5?
  - *The square roots of 5 are $\sqrt{5}$ and $-\sqrt{5}$ because $\left(\sqrt{5}\right)^2 = 5$ and $\left(-\sqrt{5}\right)^5 = 5$.*

MP.3

3. **Every real number has two square roots. Explain why.**

   The fundamental theorem of algebra guarantees that a second-degree polynomial equation has two solutions. To find the square roots of a real number b, we need to solve the equation $z^2 = b$, which is a second-degree polynomial equation. Thus, the two solutions of $z^2 = b$ are the two square roots of b. If a is one of the square roots, then $-a$ is the other.

4. **Provide a convincing argument that every complex number must also have two square roots.**

   By the same reasoning, if w is a complex number, then the polynomial equation $z^2 = w$ has two solutions. The two solutions to this quadratic equation are the square roots of w. If $a + bi$ is one square root, then $a - bi$ is the other.

MP.3

5. **Explain how the polynomial identity $x^2 - b = (x - \sqrt{b})(x + \sqrt{b})$ relates to the argument that every number has two square roots.**

   To solve $x^2 = b$, we can solve $x^2 - b = 0$. Since this quadratic equation has two distinct solutions, we can find two square roots of b. The two square roots are opposites of each other.

6. **What is the other square root of $119 + 120i$?**

   It would be the opposite of $12 + 5i$, which is the complex number, $-12 - 5i$.

Example 1 (10 minutes): Find the Square Roots of $119 + 120i$

The problem with using the polar form of a complex number to find its square roots is that the argument of these numbers is not an easily recognizable number unless we pick our values of a and b very carefully, such as $1 + \sqrt{3}i$.

Recall from the last lesson that we proved using complex numbers that the equation $x^2 = 1$ had exactly two solutions. Here is another approach to finding both square roots of a complex number that involves creating and solving a system of equations. The solutions to these equations provide a way to define the square roots of a complex number. Students have to solve a fourth-degree polynomial equation that has both real and imaginary solutions by factoring using polynomial identities.

Scaffolding:

- Students may benefit from practice and review of factoring fourth-degree polynomials. See Algebra II Module 1. Some sample problems are provided below.

$$x^4 - 2x^2 - 8$$
$$x^4 - 9x^2 - 112$$
$$x^4 - x^2 - 12$$

- Post the following identities on the board:
  $a^2 - b^2 = (a + b)(a - b)$
  $a^2 + b^2 = (a + bi)(a - bi)$.
- Students could use technology like Desmos to quickly find real number solutions to equations like those in part (b) by graphing each side and finding the x-coordinates of the intersection points.

Example: Find the Square Roots of $119 + 120i$

Let $w = p + qi$ be the square root of $119 + 120i$. Then

$$w^2 = 119 + 120i$$

and

$$(p + qi)^2 = 119 + 120i.$$

a. **Expand the left side of this equation.**

   $p^2 - q^2 + 2pqi = 119 + 120i$

b. **Equate the real and imaginary parts, and solve for p and q.**

   $p^2 - q^2 = 119$ and $2pq = 120$. Solving for q and substituting gives

$$p^2 - \left(\frac{60}{p}\right)^2 = 119.$$

   Multiplying by p^2 gives the equation

$$p^4 - 119p^2 - 3600 = 0.$$

   And this equation can be solved by factoring.

$$(p^2 + 25)(p^2 - 144) = 0$$

   We now have two polynomial expressions that we know how to factor: the sum and difference of squares.

$$(p + 5i)(p - 5i)(p - 12)(p + 12) = 0$$

   The solutions are $5i$, $-5i$, 12, and -12. Since p must be a real number by the definition of complex number, we can choose 12 or -12 for p. Using the equation $2pq = 120$, when $p = 12$, $q = 5$, and when $p = -12$, $q = -5$.

c. **What are the square roots of $119 + 120i$?**

Thus, the square roots of $119 + 120i$ are $12 + 5i$ and $-12 - 5i$.

Debrief this example by discussing how this process could be generalized for any complex number $z = a + bi$. Be sure to specifically mention that the polynomial identities (sum and difference of squares) each has two factors, so we get two solutions when setting those factors equal to zero.

- If we solved this problem again using different values of a and b instead of 119 and 120, would we still get exactly two square roots of the form $w = p + qi$ and $w = -p - qi$ that satisfy $w^2 = a + bi$? Explain your reasoning.
  - *Yes. When calculating w^2, we would get a fourth-degree equation that would factor into two second-degree polynomial factors: One that is a difference of squares, and one that is a sum of squares. The difference of squares will have two linear factors with real coefficients giving us two values of p that can be used to find the two square roots $p + qi$.*

Exercises 7–9 (12 minutes)

Students can work these exercises in groups. Be sure to have at least one group present their solution to Exercise 8. Before Exercise 7, use the discussion questions that follow if needed to activate students' prior knowledge about complex conjugates.

Exercises 7–9

7. **Use the method in the Example to find the square roots of $1 + \sqrt{3}i$.**

$p^2 - q^2 = 1$ and $2pq = \sqrt{3}$

Substituting and solving for p,

$$p^2 - \left(\frac{\sqrt{3}}{2p}\right)^2 = 1$$
$$4p^4 - 3 = 4p^2$$
$$4p^4 - 4p^2 - 3 = 0$$
$$(2p^2 - 3)(2p^2 + 1) = 0$$

gives the real solutions $p = \sqrt{\frac{3}{2}}$ or $p = -\sqrt{\frac{3}{2}}$. The values of q would then be

$q = \frac{\sqrt{3}}{2\sqrt{\frac{3}{2}}} = \frac{\sqrt{2}}{2}$ *and* $q = -\frac{\sqrt{2}}{2}$.

The square roots of $1 + \sqrt{3}i$ are $\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$ and $-\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$.

8. Find the square roots of each complex number.

a. $5+12i$

The square roots of $5+12i$ satisfy the equation $(p+qi)^2 = 5+12i$.

Expanding $(p+qi)^2$ and equating the real and imaginary parts gives

$$\begin{aligned} p^2 - q^2 &= 5 \\ 2pq &= 12. \end{aligned}$$

Substituting $q = \frac{6}{p}$ into $p^2 - q^2 = 5$ gives

$$\begin{aligned} p^2 - \left(\frac{6}{p}\right)^2 &= 5 \\ p^4 - 36 &= 5p^2 \\ p^4 - 5p^2 - 36 &= 0 \\ (p^2-9)(p^2+4) &= 0. \end{aligned}$$

The one positive real solution to this equation is 3. Let $p = 3$. Then $q = 2$. A square root of $5+12i$ is $3+2i$.

The other square root is when $p = -3$ and $q = -2$. Therefore, $-3-2i$ is the other square root of $5+12i$.

b. $5-12i$

The square roots of $5-12i$ satisfy the equation $(p+qi)^2 = 5-12i$.

Expanding $(p+qi)^2$ and equating the real and imaginary parts gives

$$\begin{aligned} p^2 - q^2 &= 5 \\ 2pq &= -12. \end{aligned}$$

Substituting $q = -\frac{6}{p}$ into $p^2 - q^2 = 5$ gives

$$\begin{aligned} p^2 - \left(-\frac{6}{p}\right)^2 &= 5 \\ p^4 - 36 &= 5p^2 \\ p^4 - 5p^2 - 36 &= 0 \\ (p^2-9)(p^2+4) &= 0. \end{aligned}$$

The one positive real solution to this equation is 3. Let $p = 3$. Then $q = -2$. A square root of $5-12i$ is $3-2i$.

The other square root is when $p = -3$ and $q = 2$. Therefore, $-3+2i$ is the other square root of $5-12i$.

- What do we call complex numbers of the form $a+bi$ and $a-bi$?
  - *They are called complex conjugates.*
- Based on Exercise 6, how are square roots of conjugates related? Why do you think this relationship exists?
  - *It appears that the square roots are opposites. When we solved the equation, the only difference was that $2pq = -12$, which ended up not mattering when we squared $-\frac{6}{p}$.*
- What is the conjugate of $119+120i$?
  - *The conjugate is $119-120i$.*

- What do you think the square roots of $119 - 120i$ would be? Explain your reasoning.
  - *The square roots of $119 - 120i$ should be the conjugates of the square roots of $119 + 120i$; the square roots should be $12 - 5i$ and $-12 + 5i$.*

9. Show that if $p + qi$ is a square root of $z = a + bi$, then $p - qi$ is a square root of the conjugate of z, $\bar{z} = a - bi$.

a. Explain why $(p + qi)^2 = a + bi$.

If $p + qi$ is a square root of $a + bi$, then it must satisfy the definition of a square root. The square root of a number raised to the second power should equal the number.

b. What do a and b equal in terms of p and q?

Expanding $(p + qi)^2 = p^2 - q^2 + 2pqi$. Thus, $a = p^2 - q^2$ and $b = 2pq$.

c. Calculate $(p - qi)^2$. What is the real part, and what is the imaginary part?

$(p - qi)^2 = p^2 - q^2 - 2pqi$ The real part is $p^2 - q^2$, and the imaginary part is $-2pq$.

d. Explain why $(p - qi)^2 = a - bi$.

From part (c), $p^2 - q^2 = a$ and $2pq = b$. Substituting,

$(p - qi)^2 = p^2 - q^2 - 2pqi = a - bi$.

Closing (3 minutes)

Students can respond to the following questions either in writing or by discussing them with a partner.

- How are square roots of complex numbers found by solving a polynomial equation?
  - *If $p + qi$ is a square root of a complex number $a + bi$, then $(p + qi)^2 = a + bi$. Expanding the expression on the left and equating the real and imaginary parts leads to equations $p^2 - q^2 = a$ and $2pq = b$. Then, solve this resulting system of equations for p and q.*
- Explain why the fundamental theorem of algebra guarantees that every complex number has two square roots.
  - *According to the fundamental theorem of algebra, every second-degree polynomial equation factors into linear terms. The square roots of a complex number w are solutions to the equation $z^2 = w$, which can be rewritten as $z^2 - w = 0$. This equation has two solutions, each of which is a square root of w.*

Lesson Summary

The square roots of a complex number $a + bi$ are of the form $p + qi$ and $-p - qi$ and can be found by solving the equations $p^2 - q^2 = a$ and $2pq = b$.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 2: Does Every Complex Number Have a Square Root?

Exit Ticket

1. Find the two square roots of $5 - 12i$.

2. Find the two square roots of $3 - 4i$.

Exit Ticket Sample Solutions

1. Find the two square roots of $5 - 12i$.

   $(p + qi)^2 = 5 - 12i$, $p^2 - q^2 + 2pqi = 5 - 12i$,

   $p^2 - q^2 = 5$, $pq = -6$, $q = -\frac{6}{p}$

   $p^2 - \frac{36}{p^2} - 5 = 0$, $p^4 - 5p^2 - 36 = 0$, $(p^2 + 4)(p^2 - 9) = 0$, $p = \pm 3$,

   $p = 3$, $q = -2$; $p = -3$, $q = 2$

   Therefore, the square roots are $3 - 2i$ *and* $-3 + 2i$.

2. Find the two square roots of $3 - 4i$.

   Let the square roots have the form $p + qi$. *Then* $(p + qi)^2 = 3 - 4i$. *This gives* $p^2 - q^2 = 3$ *and* $2pq = -4$. *Then* $p = -\frac{2}{q}$, *so* $p^2 - q^2 = \frac{4}{q^2} - q^2 = 3$, *and* $q^4 + 3q^2 - 4 = 0$. *This expression factors into* $(q + 1)(q - 1)(q + 2i)(q - 2i) = 0$, *and we see that the only real solutions are* $q = 1$ *and* $q = -1$. *If* $q = 1$, *then* $p = -2$, *and if* $q = -1$, *then* $p = 2$. *Thus, the two square roots of* $3 - 4i$ *are* $-2 + i$ *and* $2 - i$.

Problem Set Sample Solutions

Find the two square roots of each complex number by creating and solving polynomial equations.

1. $z = 15 - 8i$

   $(p + qi)^2 = 15 - 8i$, $p^2 - q^2 + 2pqi = 15 - 8i$,

   $p^2 - q^2 = 15$, $pq = -4$, $q = -\frac{4}{p}$

   $p^2 - \frac{16}{p^2} - 15 = 0$, $p^4 - 15p^2 - 16 = 0$, $(p^2 + 1)(p^2 - 16) = 0$, $p = \pm 4$,

   $p = 4$, $q = -1$; $p = -4$, $q = 1$

   Therefore, the square roots of $15 - 8i$ *are* $4 - i$ *and* $-4 + i$.

2. $z = 8 - 6i$

   $(p + qi)^2 = 8 - 6i$, $p^2 - q^2 + 2pqi = 8 - 6i$,

   $p^2 - q^2 = 8$, $pq = -3$, $q = -\frac{3}{p}$

   $p^2 - \frac{9}{p^2} - 8 = 0$, $p^4 - 8p^2 - 9 = 0$, $(p^2 + 1)(p^2 - 9) = 0$, $p = \pm 3$,

   $p = 3$, $q = -1$; $p = -3$, $q = 1$

   Therefore, the square roots of $8 - 6i$ *are* $3 - i$ *and* $-3 + i$.

3. $z = -3 + 4i$

$(p + qi)^2 = -3 + 4i$, $p^2 - q^2 + 2pqi = -3 + 4i$,

$p^2 - q^2 = -3$, $pq = 2$, $q = \frac{2}{p}$

$p^2 - \frac{4}{p^2} + 3 = 0$, $p^4 + 3p^2 - 4 = 0$, $(p^2 + 4)(p^2 - 1) = 0$, $p = \pm 1$,

$p = 1$, $q = 2$; $p = -1$, $q = -2$

Therefore, the square roots of $-3 + 4i$ *are* $1 + 2i$ *and* $-1 - 2i$.

4. $z = -5 - 12i$

$(p + qi)^2 = -5 - 12i$, $p^2 - q^2 + 2pqi = -5 - 12i$,

$p^2 - q^2 = -5$, $pq = -6$, $q = \frac{-6}{p}$

$p^2 - \frac{6}{p^2} + 5 = 0$, $p^4 + 5p^2 - 36 = 0$, $(p^2 + 9)(p^2 - 4) = 0$, $p = \pm 2$,

$p = 2$, $q = -3$; $p = -2$, $q = 3$

Therefore, the square roots of $-5 - 12i$ *are* $2 - 3i$ *and* $-2 + 3i$.

5. $z = 21 - 20i$

$(p + qi)^2 = 21 - 20i$, $p^2 - q^2 + 2pqi = 21 - 20i$,

$p^2 - q^2 = 21$, $pq = -10$, $q = -\frac{10}{p}$

$p^2 - \frac{100}{p^2} - 21 = 0$, $p^4 - 21p^2 - 100 = 0$, $(p^2 + 4)(p^2 - 25) = 0$, $p = \pm 5$,

$p = 5$, $q = -2$; $p = -5$, $q = 2$

Therefore, the square roots of $21 - 20i$ *are* $5 - 2i$ *and* $-5 + 2i$.

6. $z = 16 - 30i$

$(p + qi)^2 = 16 - 30i$, $p^2 - q^2 + 2pqi = 16 - 30i$,

$p^2 - q^2 = 16$, $pq = -15$, $q = -\frac{15}{p}$

$p^2 - \frac{225}{p^2} - 16 = 0$, $p^4 - 16p^2 - 225 = 0$, $(p^2 + 9)(p^2 - 25) = 0$, $p = \pm 5$,

$p = 5$, $q = -3$; $p = -5$, $q = 3$

Therefore, the square roots of $16 - 30i$ *are* $5 - 3i$ *and* $-5 + 3i$.

7. $z = i$

$(p + qi)^2 = 0 + i,\ p^2 - q^2 + 2pqi = 0 + i,$

$p^2 - q^2 = 0,\ pq = \frac{1}{2},\ q = \frac{1}{2p}$

$p^2 - \frac{1}{4p^2} = 0,\ 4p^4 - 1 = 0,\ (2p^2 + 1)(2p^2 - 1) = 0,\ p = \pm\frac{\sqrt{2}}{2}$

$p = \frac{\sqrt{2}}{2},\ q = \frac{\sqrt{2}}{2};\ p = -\frac{\sqrt{2}}{2},\ q = -\frac{\sqrt{2}}{2}$

Therefore, the square roots of i are $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ *and* $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

A *Pythagorean triple* is a set of three positive integers a, b, and c such that $a^2 + b^2 = c^2$. Thus, these integers can be the lengths of the sides of a right triangle.

8. **Show algebraically that for positive integers p and q, if**

$$a = p^2 - q^2$$
$$b = 2pq$$
$$c = p^2 + q^2$$

then $a^2 + b^2 = c^2$.

$$\begin{aligned} a^2 + b^2 &= (p^2 - q^2)^2 + (2pq)^2 \\ &= p^4 - 2p^2q^2 + q^4 + 4p^2q^2 \\ &= p^4 + 2p^2q^2 + q^4 \\ &= p^4 + 2p^2q^2 + q^4 \\ &= c^2 \end{aligned}$$

9. **Select two integers p and q, use the formulas in Problem 8 to find a, b, and c, and then show those numbers satisfy the equation $a^2 + b^2 = c^2$.**

Let $p = 3$ and $q = 2$. Calculate the values of a, b, and c.

$$a = 3^2 - 2^2 = 5$$

$$b = 2(3)(2) = 12$$

$$c = 3^2 + 2^2 = 13$$

$$a^2 + b^2 = 5^2 + 12^2 = 25 + 144 = 169 = 13^2 = c^2$$

10. **Use the formulas from Problem 8, and find values for p and q that give the following famous triples.**

a. $(3, 4, 5)$

$a = p^2 - q^2 = 3,\ b = 2pq = 4,\ c = p^2 + q^2 = 5$

$2p^2 = 8,\ p = 2,\ q = 1$

$(p^2 - q^2)^2 + (2pq)^2 = (p^2 - q^2)^2$

$(2^2 - 1^2)^2 + \left(2(2)(1)\right)^2 = (2^2 + 1^2)^2$

$(4 - 1)^2 + 4^2 = (4 + 1)^2,\ (3)^2 + (4)^2 = (5)^2$

b. $(5, 12, 13)$

$a = p^2 - q^2 = 5, b = 2pq = 12, c = p^2 + q^2 = 13$

$2p^2 = 18,\ p = 3, q = 2$

$(p^2 - q^2)^2 + (2pq)^2 = (p^2 - q^2)^2$

$(3^2 - 2^2)^2 + \big(2(3)(2)\big)^2 = (3^2 + 2^2)^2$

$(9 - 4)^2 + 12^2 = (9 + 4)^2, (5)^2 + (12)^2 = (13)^2$

c. $(7, 24, 25)$

$a = p^2 - q^2 = 7, b = 2pq = 24, c = p^2 + q^2 = 25,$

$2p^2 = 32, p = 4, q = 3$

$(p^2 - q^2)^2 + (2pq)^2 = (p^2 - q^2)^2$

$(4^2 - 3^2)^2 + \big(2(4)(3)\big)^2 = (4^2 + 3^2)^2$

$(16 - 9)^2 + 24^2 = (16 + 9)^2, (7)^2 + (24)^2 = (25)^2$

d. $(9, 40, 41)$

$a = p^2 - q^2 = 9, b = 2pq = 40, c = p^2 + q^2 = 41,$

$2p^2 = 50, p = 5, q = 4$

$(p^2 - q^2)^2 + (2pq)^2 = (p^2 - q^2)^2$

$(5^2 - 4^2)^2 + \big(2(5)(4)\big)^2 = (5^2 + 4^2)^2$

$(25 - 16)^2 + 40^2 = (25 + 16)^2, (9)^2 + (40)^2 = (41)^2$

11. **Is it possible to write the Pythagorean triple $(6, 8, 10)$ in the form $a = p^2 - q^2$, $b = 2pq$, $c = p^2 + q^2$ for some integers p and q? Verify your answer.**

$a = p^2 - q^2 = 6, b = 2pq = 8, c = p^2 + q^2 = 10,$

$2p^2 = 16, p^2 = 8, p = \pm 2\sqrt{2}, q = \pm\sqrt{2}$

The Pythagorean triple $(6, 8, 10)$ cannot be written in the form $a = p^2 - q^2$, $b = 2pq$, $c = p^2 + q^2$, for any integers p and q.

12. **Choose your favorite Pythagorean triple (a, b, c) that has a and b sharing only 1 as a common factor, for example $(3, 4, 5)$, $(5, 12, 13)$, or $(7, 24, 25)$, … Find the square of the length of a square root of $a + bi$; that is, find $|p + qi|^2$, where $p + qi$ is a square root of $a + bi$. What do you observe?**

For $(3, 4, 5)$, $a = 3$, $b = 4$, $a + bi = (p + qi)^2 = (-p - qi)^2$,

$a + bi = 3 + 4i = (p + qi)^2 = p^2 - q^2 + 2pqi$; *therefore,* $p^2 - q^2 = 3$, $2pq = 4$, $q = \frac{2}{p}$

$p^2 - \frac{4}{p^2} = 3, p^4 - 3p^2 - 4 = 0, (p^2 + 1)(p^2 - 4) = 0, p = \pm 2$

$p = 2, q = 1$; *therefore,* $p + qi = 2 + i$.

$p = -2, q = -1$; *therefore,* $p + qi = -2 - i$.

The two square roots of $a + bi$ are $2 + i$ and $-2 - i$. Both of these have length $\sqrt{5}$, so the squared length is 5. This is the third value c in the Pythagorean triple (a, b, c).

Lesson 3: Roots of Unity

Student Outcomes

- Students determine the complex roots of polynomial equations of the form $x^n = 1$ and, more generally, equations of the form $x^n = k$ positive integers n and positive real numbers k.
- Students plot the n^{th} roots of unity in the complex plane.

Lesson Notes

This lesson ties together work from Algebra II Module 1, where students studied the nature of the roots of polynomial equations and worked with polynomial identities and their recent work with the polar form of a complex number to find the n^{th} roots of a complex number in Precalculus Module 1 Lessons 18 and 19. The goal here is to connect work within the algebra strand of solving polynomial equations to the geometry and arithmetic of complex numbers in the complex plane. Students determine solutions to polynomial equations using various methods and interpreting the results in the real and complex plane. Students need to extend factoring to the complex numbers (**N-CN.C.8**) and more fully understand the conclusion of the fundamental theorem of algebra (**N-CN.C.9**) by seeing that a polynomial of degree n has n complex roots when they consider the roots of unity graphed in the complex plane. This lesson helps students cement the claim of the fundamental theorem of algebra that the equation $x^n = 1$ should have n solutions as students find all n solutions of the equation, not just the obvious solution $x = 1$. Students plot the solutions in the plane and discover their symmetry.

GeoGebra can be a powerful tool to explore these types of problems and really helps students to see that the roots of unity correspond to the vertices of a polygon inscribed in the unit circle with one vertex along the positive real axis.

Classwork

Opening Exercise (3 minutes)

Form students into small groups of 3–5 students each depending on the size of the classroom. Much of this activity is exploration. Start the conversation by having students discuss and respond to the exercises in the opening. Part (c) is an important connection to make for students. More information on the fundamental theorem of algebra can be found in Algebra II Module 1 Lessons 38–40. This information is also reviewed in Lesson 1 of this module. The amount that students recall as they work on these exercises with their groups can inform decisions about how much scaffolding is necessary as students work through the rest of this lesson. If students are struggling to make sense of the first two problems, ask them to quickly find real number solutions to these equations by inspection: $x = 1$, $x^2 = 1$, $x^3 = 1$, and $x^4 = 1$.

Scaffolding:

At this level of mathematics, students may struggle to remember formulas or theorems from previous grades. A quick reference sheet or anchor chart can come in handy. The Lesson Summary boxes from the following Algebra II and Precalculus lessons would be helpful to have handy during this lesson:

- Precalculus Module 1 Lessons 13, 18, 19
- Algebra II Module 1 Lessons 6, 40

MP.3

Opening Exercise

Consider the equation $x^n = 1$ for positive integers n.

a. **Must an equation of this form have a real solution? Explain your reasoning.**

The number 1 will always be a solution to $x^n = 1$ because $1^n = 1$ for any positive integer n.

b. **Could an equation of this form have two real solutions? Explain your reasoning.**

When n is an even number, both 1 and -1 are solutions. The number 1 is a solution because $1^n = 1$ for any positive integer. The number -1 is a solution because $(-1)^n = (-1)^{2k}$ where k is a positive integer if n is even and

$$(-1)^{2k} = ((-1)^2)^k = 1^k = 1.$$

c. **How many complex solutions are there for an equation of this form? Explain how you know.**

We can rewrite the equation in the form $x^n - 1 = 0$. The solutions to this polynomial equation are the roots of the polynomial $p(x) = x^n - 1$. The fundamental theorem of algebra says that the polynomial $p(x) = x^n - 1$ factors over the complex numbers into the product of n linear terms. Each term identifies a complex root of the polynomial. Thus, a polynomial equation of degree n has at most n solutions.

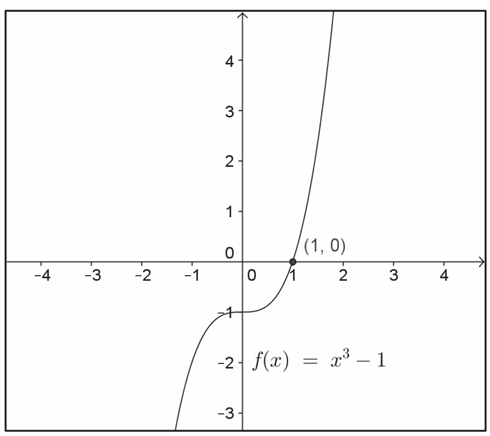
Exploratory Challenge (10 minutes)

In this Exploratory Challenge, students should work to apply multiple methods to find the solutions to the equation $x^3 = 1$. Give teams sufficient time to consider more than one method. When debriefing the solution methods students found, make sure to present and discuss the second method below, especially if most groups did not attempt the problem.

Exploratory Challenge

Consider the equation $x^3 = 1$.

a. **Use the graph of $f(x) = x^3 - 1$ to explain why 1 is the only real number solution to the equation $x^3 = 1$.**



From the graph, you can see that the point $(1, 0)$ is the x-intercept of the function. That means that 1 is a zero of the polynomial function and thus is a solution to the equation $x^3 - 1 = 0$.

b. **Find all of the complex solutions to the equation $x^3 = 1$. Come up with as many methods as you can for finding the solutions to this equation.**

Method 1: Factoring Using a Polynomial Identity and Using the Quadratic Formula

Rewrite the equation in the form $x^3 - 1 = 0$, and use the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ to factor $x^3 - 1$.

$$x^3 - 1 = 0$$
$$(x-1)(x^2 + x + 1) = 0$$

Then

$x - 1 = 0$ or $x^2 + x + 1 = 0$.

The solution to the equation $x - 1 = 0$ is 1. The quadratic formula gives the other solutions.

$$x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

So, the solution set is

$$\left\{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\right\}.$$

Method 2: Using the Polar Form of a Complex Number

The solutions to the equation $x^3 = 1$ are the cube roots of 1.

The number 1 has modulus 1 and argument 0 (or any rotation that terminates along the positive real axis such as 2π or 4π, etc.).

The modulus of the cube roots of 1 is $\sqrt[3]{1} = 1$. The arguments are solutions to

$$3\theta = 0$$
$$3\theta = 2\pi$$
$$3\theta = 4\pi$$
$$3\theta = 6\pi.$$

The solutions to these equations are $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi, \ldots$. Since the rotations cycle back to the same locations in the complex plane after the first three, we only need to consider 0, $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$.

The solutions to the equation $x^3 = 1$ are

$$1(\cos(0) + i\sin(0)) = 1$$
$$1\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$1\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Method 3: Using the Techniques of Lessons 1 and 2 from This Module

Let $x = a + bi$; then $(a + bi)^3 = 1 + 0i$.

Expand $(a + bi)^3$, and equate the real and imaginary parts with 1 and 0.

$$(a + bi)^3 = (a + bi)(a^2 + 2abi - b^2) = a^3 + 3a^2bi - 3ab^2 - b^3i$$

The real part of $(a + bi)^3$ is $a^3 - 3ab^2$, and the imaginary part is $3a^3b - b^3$. Thus, we need to solve the system

$$a^3 - 3ab^2 = 1$$

$$3a^2b - b^3 = 0.$$

Rewriting the second equation gives us

$$b(3a^2 - b^2) = 0.$$

If $b = 0$, then $a^3 = 1$ and $a = 1$. So, a solution to the equation $x^3 = 1$ is $1 + 0i = 1$.

If $3a^2 - b^2 = 0$, then $b^2 = 3a^2$, and by substitution,

$$\begin{aligned} a^3 - 3a(3a^2) &= 1 \\ -8a^3 &= 1 \\ a^3 &= -\frac{1}{8}. \end{aligned}$$

This equation has one real solution: $-\frac{1}{2}$. If $= -\frac{1}{2}$, then $b^2 = 3\left(-\frac{1}{2}\right)^2 = \frac{3}{4}$, so $b = \frac{\sqrt{3}}{2}$ or $= -\frac{\sqrt{3}}{2}$. The other two solutions to the equation $x^3 = 1$ are $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Have different groups present their solutions. If no groups present Method 2, share that with the class. The teacher may also review the formula derived in Module 1, Lesson 19 that is shown below.

Given a complex number z with modulus r and argument θ, the n^{th} roots of z are given by

$$\sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right)$$

for integers k and n such that $n > 0$ and $0 \le k < n$.

Present the next few questions giving students time to discuss each one in their groups before asking for whole-class responses.

- Which methods are you most inclined to use and why?
  - *Solving using factoring and the quadratic formula is the easiest way to do this provided you know the polynomial identity for the difference of two cubes.*
- What are some potential limitations to each method?
  - *It can be difficult to solve equations by factoring as the value of n gets largers. If $n = 5$, then we can't really factor $x^5 - 1$ beyond $(x - 1)(x^4 + x^3 + x^2 + x + 1)$ easily. The other algebraic method using $x = a + bi$ is also challenging as the value of n increases. The polar form method works well if you know your unit circle and the proper formulas and definitions.*
- Which of these approaches is the easiest to use for positive integers $n > 3$ in the equation $x^n = 1$?
  - *If you know the proper formulas and relationships between powers of complex numbers in polar form, working with the polar form of the complex solutions would be the easiest approach as n gets larger.*

Exercises 1–4 (15 minutes)

Next, read and discuss the definition of the roots of unity.

- Why do you think the solutions are called *the roots of unity*?
  - *Because the word unity implies the number* 1. *It is also like the unit circle, which has a radius of* 1.

Have students work on the next three exercises. If time is running short, assign each group only one of the exercises, but make sure at least one group is working on each one. Have them present their solutions on the board as they finish to check for errors and to prepare for Exercise 4.

Exercises

Solutions to the equation $x^n = 1$ for positive integers n are called *the n^{th} roots of unity*.

1. **What are the square roots of unity in rectangular and polar form?**

   The square roots of unity in rectangular form are the real numbers 1 and -1.
   In polar form, $1(\cos(0) + i\sin(0))$ and $1(\cos(\pi) + i\sin(\pi))$.

2. **What are the fourth roots of unity in rectangular and polar form? Solve this problem by creating and solving a polynomial equation. Show work to support your answer.**

   The fourth roots of unity in rectangular form are $1, -1, i, -i$.

$$x^4 = 1$$
$$x^4 - 1 = 0$$
$$(x^2-1)(x^2+1) = 0$$

   The solutions to $x^2 - 1 = 0$ are 1 and -1. The solutions to $x^2 + 1 = 0$ are i and $-i$.
   In polar form, $1(\cos(0) + i\ \sin(0))$, $1\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$, $1(\cos(\pi) + i\sin(\pi))$, $1\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$.

3. **Find the sixth roots of unity in rectangular form by creating and solving a polynomial equation. Show work to support your answer. Find the sixth roots of unity in polar form.**

$$x^6 - 1 = 0$$
$$(x^3+1)(x^3-1) = 0$$
$$(x+1)(x^2-x+1)(x-1)(x^2+x+1) = 0$$

   By inspection, 1 and -1 are sixth roots. Using the quadratic formula to find the solutions to $x^2 - x + 1 = 0$ and $x^2 + x + 1 = 0$ gives the other four roots: $\frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}$, and $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$.
   In polar form, $1(\cos(0) + i\sin(0))$, $1\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$, $\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$, $1(\cos(\pi) + i\sin(\pi))$, $1\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$, $1\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$.

MP.8

Start a chart on the board like the one shown below. As groups finish, have them record their responses on the chart. The teacher or volunteer students can record the polar forms of these numbers. Notice that the fifth roots of unity cannot be written as easily recognizable numbers in rectangular form. Ask students to look at the patterns in this table as they finish their work and begin to make a generalization about the fifth roots of unity.

MP.8

n	n^{th} roots of unity in rectangular form	n^{th} roots of unity in polar form
2	1 and -1	$1(\cos(0) + i\sin(0))$ and $1(\cos(\pi) + i\sin(\pi))$
3	$1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$	$1(\cos(0) + i\sin(0))$ $1\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$ $1\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$
4	$1, -1, i, -i$	$1(\cos(0) + i\sin(0))$ $1\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ $1(\cos(\pi) + i\sin(\pi))$ $1\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$
5	1	
6	$1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$ $-1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$	$1(\cos(0) + i\sin(0))$ $1\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$ $1\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$ $1(\cos(\pi) + i\sin(\pi))$ $1\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$ $1\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$

4. **Without using a formula, what would be the polar forms of the fifth roots of unity? Explain using the geometric effect of multiplication complex numbers.**

The modulus would be 1 because dividing 1 into the product of six equal numbers still means each number must be 1. The arguments would be fifths of 2π, so $0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}$, and $\frac{4\pi}{5}$. The fifth roots of z, when multiplied together, must equal $z^1 = z^{\frac{1}{5}} \cdot z^{\frac{1}{5}} \cdot z^{\frac{1}{5}} \cdot z^{\frac{1}{5}} \cdot z^{\frac{1}{5}}$. That would be like starting with the real number 1 and rotating it by $\frac{1}{5}$ of 2π and dilating it by a factor of 1 so that you ended up back at the real number 1 after 5 repeated multiplications.

If students are struggling to understand the geometric approach, the teacher may also use the formula developed in Lesson 19. Early finishers could be asked to verify that the formula also provides the correct roots of unity.

The modulus is 1 because $\sqrt[5]{1} = 1$. For $k = 0$ to 4, the arguments are

$$\frac{0}{5} + \frac{2\pi \cdot 0}{5} = 0$$
$$\frac{0}{5} + \frac{2\pi \cdot 1}{5} = \frac{2\pi}{5}$$
$$\frac{0}{5} + \frac{2\pi \cdot 2}{5} = \frac{4\pi}{5}$$
$$\frac{0}{5} + \frac{2\pi \cdot 3}{5} = \frac{6\pi}{5}$$
$$\frac{0}{5} + \frac{2\pi \cdot 4}{5} = \frac{8\pi}{5}.$$

The fifth roots of unity are

$$1(\cos(0) + i\sin(0))$$
$$1\left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right)$$
$$1\left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right)$$
$$1\left(\cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)\right)$$
$$1\left(\cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)\right).$$

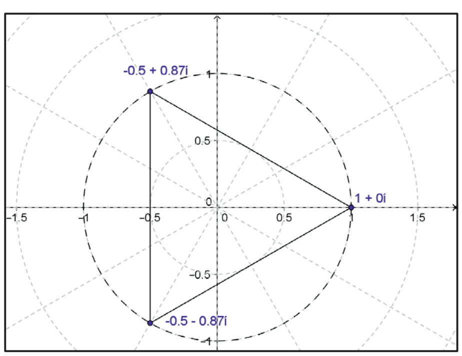
Discussion (8 minutes)

Display the roots of unity for $n > 2$, either by using GeoGebra or by showing the diagrams below. To create these graphics in GeoGebra, enter the complex number $z = 1 + 0i$. Then, rotate this point about the origin by $\frac{2\pi}{n}$ to get the next root of unity, and then rotate that point about the origin by $\frac{2\pi}{n}$ to get the next root of unity, etc. Then, draw segments connecting adjacent points. Ask students to discuss their observations in small groups. Have them summarize their responses below each question.

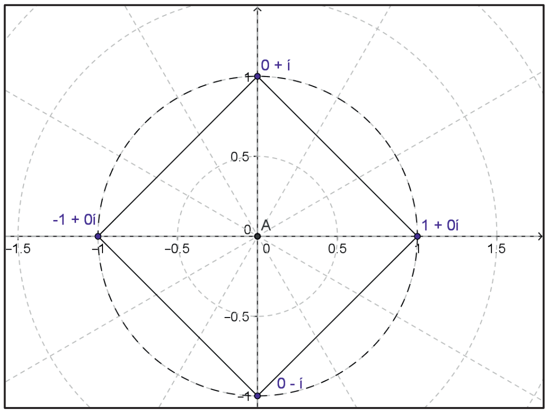
Scaffolding:

- For students that need a more concrete approach, provide polar grid paper, and have them plot the roots of unity by hand.
- Have students plot the roots of unity on their own using GeoGebra if technology is available. Encourage them to use the transformations menu to plot the roots of unity as rotations of $1 + 0i$.

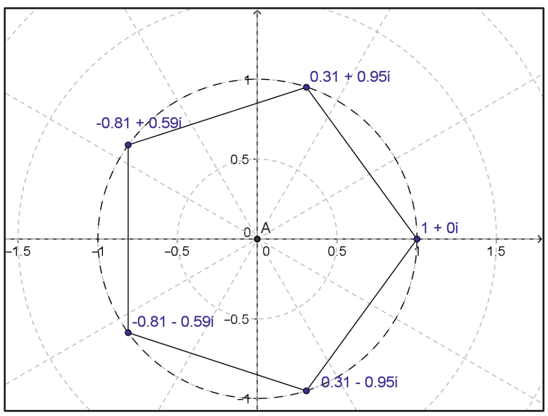
The Cube Roots of Unity. The numbers are in rectangular form.



The Fourth Roots of Unity



The Fifth Roots of Unity



MP.7 & MP.8

Discussion

What is the modulus of each root of unity regardless of the value of n? Explain how you know.

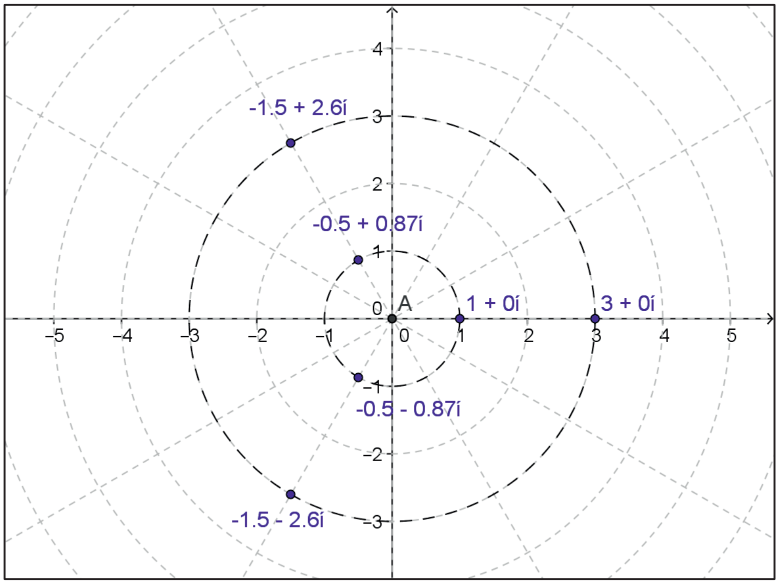
The modulus is always 1 because the n^{th} root of 1 is equal to 1. The points are on the unit circle, and the radius is always 1.

How could you describe the location of the roots of unity in the complex plane?

They are points on a unit circle, evenly spaced every $\frac{2\pi}{n}$ units starting from 1 along the positive real axis.

MP.7 & MP.8

The diagram below shows the solutions to the equation $x^3 = 27$. How do these numbers compare to the cube roots of unity (e.g., the solutions to $x^3 = 1$)?



They are points on a circle of radius 3 since the cube root of 27 is 3. Each one is a scalar multiple (by a factor of 3) of the cube roots of unity. Thus, they have the same arguments but a different modulus.

Closing (5 minutes)

Use the questions below to help students process the information in the Lesson Summary. They can respond individually or with a partner.

- What are the n^{th} roots of unity?
  - *They are the n complex solutions to an equation of the form $x^n = 1$, where n is a positive integer.*
- How can we tell how many real solutions an equation of the form $x^n = 1$ or $x^n = k$ for integers n and positive real numbers k has? How many complex solutions are there?
  - *The real number 1 (or $\sqrt[n]{k}$) is a solution to $x^n = 1$ (or $x^n = k$) regardless of the value of n. The real number -1 (or $-\sqrt[n]{k}$) is also a solution when n is an even number. The fundamental theorem of algebra says that a degree n polynomial function has n roots, so these equations have n complex solutions.*
- How can the polar form of a complex number and the geometric effect of complex multiplication help to find all the complex solutions to equations of the form $x^n = 1$ and $x^n = k$ for positive integers n and a positive real number k?
  - *One solution to $x^n = 1$ is always the complex number $1 + 0i$. Every other solution can be obtained by $n - 1$ subsequent rotations of the complex number clockwise about the origin by $\frac{2\pi}{n}$ radians. For the equation $x^n = k$, there are n solutions with modulus $\sqrt[n]{k}$ and arguments of $\left\{0, \frac{2\pi}{n}, \frac{4\pi}{n}, \frac{2\pi}{n}, \dots, \frac{2\pi(n-1)}{n}\right\}$.*

Lesson Summary

The solutions to the equation $x^n = 1$ for positive integers n are called the n^{th} roots of unity. For any value of $n > 2$, the roots of unity are complex numbers of the form $z_k = a_k + b_k i$ for positive integers $1 < k < n$ with the corresponding points (a_k, b_k) at the vertices of a regular n-gon centered at the origin with one vertex at $(1, 0)$.

The fundamental theorem of algebra guarantees that an equation of the form $x^n = k$ will have n complex solutions. If n is odd, then the real number $\sqrt[n]{k}$ is the only real solution. If n is even, then the equation has exactly two real solutions: $\sqrt[n]{k}$ and $-\sqrt[n]{k}$.

Given a complex number z with modulus r and argument θ, the n^{th} roots of z are given by

$$\sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right)$$

for integers k and n such that $n > 0$ and $0 \leq k < n$.

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 3: Roots of Unity

Exit Ticket

1. What is a fourth root of unity? How many fourth roots of unity are there? Explain how you know.

2. Find the polar form of the fourth roots of unity.

3. Write $x^4 - 1$ as a product of linear factors, and explain how this expression supports your answers to Problems 1 and 2.

Exit Ticket Sample Solutions

1. **What is a fourth root of unity? How many fourth roots of unity are there? Explain how you know.**

   The fourth root of unity is a number that multiplied by itself 4 times is equal to 1. There are 4 fourth roots of unity. $x^4 = 1$ results in solving the polynomial $x^4 - 1 = 0$. The fundamental theorem of algebra guarantees four roots since that is the degree of the polynomial.

2. **Find the polar form of the fourth roots of unity.**

   For the fourth roots of unity, $n = 4$ and $r = 1$, so each root has modulus 1, and the arguments are $0, \frac{2\pi}{4}, \frac{4\pi}{4}$, and $\frac{6\pi}{4}$. We can rewrite the arguments as $0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$. Then, the fourth roots of unity are

$$x_1 = \cos(0) + i\sin(0) = 1$$
$$x_2 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$
$$x_3 = \cos(\pi) + i\sin(\pi) = -1$$
$$x_4 = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i.$$

3. **Write $x^4 - 1$ as a product of linear factors, and explain how this expression supports your answers to Problems 1 and 2.**

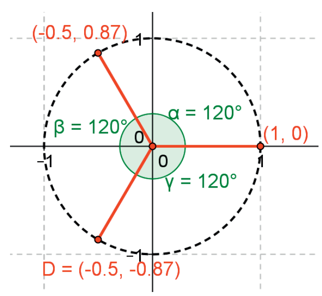
   Since there are four roots of unity, there should be four linear factors.

$$x^4 - 1 = (x-1)(x-i)(x+1)(x+i)$$

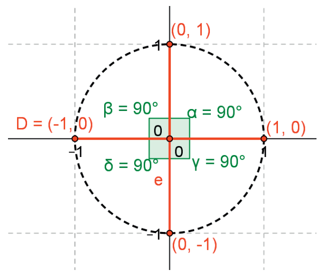
Problem Set Sample Solutions

1. **Graph the n^{th} roots of unity in the complex plane for the specified value of n.**

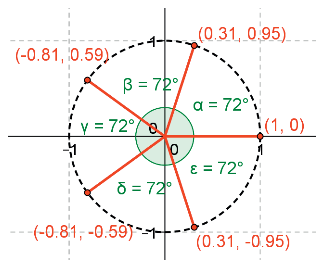
   a. $n = 3$



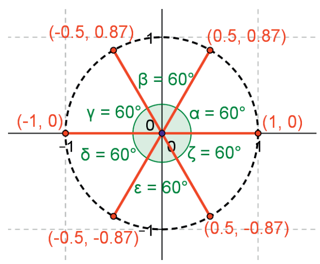
b. $n = 4$



c. $n = 5$



d. $n = 6$



2. **Find the cube roots of unity by using each method stated.**

a. **Solve the polynomial equation $x^3 = 1$ algebraically.**

$x^3 - 1 = 0,\ (x-1)(x^2+x+1) = 0,\ x = 1,\ x^2+x+1 = 0,\ x = \frac{-1\pm\sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$

The roots of unity are $1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$

b. Use the polar form $z^3 = r(\cos(\theta) + \sin(\theta))$, and find the modulus and argument of z.

$z^3 = 1$, $r^3 = 1$, $r = 1$

$3\theta = 0$, $3\theta = 2\pi$, $3\theta = 4\pi$, …; *therefore,* $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi, \ldots$

$z = \sqrt[3]{r}(\cos(\theta) + i\sin(\theta))$

$z_1 = 1(\cos(0) + i\sin(0)) = 1$

$z_2 = 1\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_3 = 1\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

The roots of unity are $1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

c. Solve $(a + bi)^3 = 1$ by expanding $(a + bi)^3$ and setting it equal to $1 + 0i$.

$(a + bi)^3 = 1$, $a^3 + 3a^2bi - 3ab^2 - b^3i = 1$; *therefore,* $a^3 - 3ab^2 = 1$ *and* $3a^2b - b^3 = 0$.

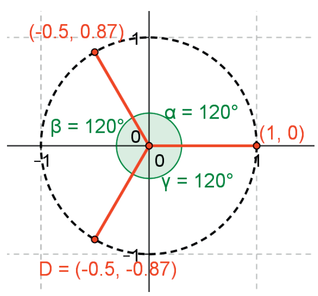
For $3a^2b - b^3 = 0$, $b(3a^2 - b^2) = 0$, *we have either* $b = 0$ *or* $a^2 - b^2 = 0$.

For $b = 0$, *we substitute it in* $a^3 - 3ab^2 = 1$, $a^3 = 1$, $a = 1$; *therefore, we have* $1 + 0i$.

For $3a^2 - b^2 = 0$, $b^2 = 3a^2$, *we substitute it in* $a^3 - 3ab^2 = 1$, $a^3 - 9a^3 = 1$, $a^3 = -\frac{1}{8}$, $a = -\frac{1}{2}$.

For $= -\frac{1}{2}$, *we substitute it in* $b^2 = 3a^2$, *and we get* $b = \pm\frac{\sqrt{3}}{2}$. *Therefore, we have* $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ *and* $\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

The roots of unity are $1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.



3. Find the fourth roots of unity by using the method stated.

a. Solve the polynomial equation $x^4 = 1$ algebraically.

$x^4 - 1 = 0$, $(x^2 + 1)(x + 1)(x - 1) = 0$, $x = \pm i$, $x = \pm 1$

The roots of unity are $1, i, -1, -i$.

b. Use the polar form $z^4 = r(\cos(\theta) + \sin(\theta))$, and find the modulus and argument of z.

$z^4 = 1$, $r^4 = 1$, $r = 1$

$4\theta = 0$, $4\theta = 2\pi$, $4\theta = 4\pi$, $4\theta = 6\pi$, $4\theta = 8\pi$, …; *therefore,* $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \ldots$

$z = \sqrt[3]{r}(\cos(\theta) + i\sin(\theta))$

$z_1 = 1(\cos(0) + i\sin(0)) = 1$

$z_2 = 1\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = i$

$z_3 = 1(\cos(\pi) + i\sin(\pi)) = -1$

$z_4 = 1\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right) = -i$

The roots of unity are $1, i, -1, -i$.

c. Solve $(a + bi)^4 = 1$ by expanding $(a + bi)^4$ and setting it equal to $1 + 0i$.

$(a + bi)^4 = 1$, $a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4 = 1$.

Therefore, $a^4 - 6a^2b^2 + b^4 = 1$ *and* $4a^3b - 4ab^3 = 0$.

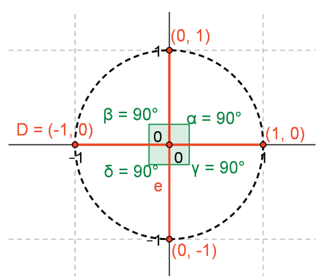
For $4a^3b - 4ab^3 = 0$, $4ab(a^2 - b^2) = 0$, *we have either* $a = 0$, $b = 0$, *or* $a^2 - b^2 = 0$.

For $a = 0$, *we substitute it in* $a^4 - 6a^2b^2 + b^4 = 1$, $b^4 = 1$, $b = \pm 1$; *therefore, we have fourth roots of unity* i *and* $-i$.

For $b = 0$, *we substitute it in* $a^4 - 6a^2b^2 + b^4 = 1$, $a^4 = 1$, $a = \pm 1$ *for* a. $b \in \mathbb{R}$; *therefore, we have fourth roots of unity* 1 *and* -1.

For $a^2 - b^2 = 0$, $a^2 = b^2$, *we substitute it in* $a^4 - 6a^2b^2 + b^4 = 1$, $b^4 - 6b^4 + b^4 = 1$, $4b^4 = -1$; *there is no solution for* b *for* a. $b \in \mathbb{R}$;

The roots of unity are $1, i, -1, -i$.



4. **Find the fifth roots of unity by using the method stated.**

Use the polar form $z^5 = r(\cos(\theta) + \sin(\theta))$, and find the modulus and argument of z.

$z^5 = 1$, $r^5 = 1$, $r = 1$

$5\theta = 0$, $5\theta = 2\pi$, $5\theta = 4\pi$, $5\theta = 6\pi$, $5\theta = 8\pi$, $5\theta = 10\pi$, …; *therefore,* $\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, 2\pi, \ldots$

$z_1 = 1(\cos(0) + i\ \sin(0)) = 1$

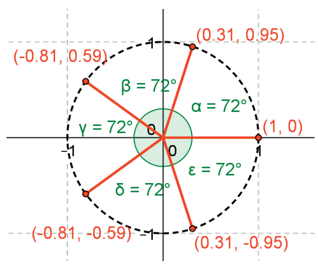
$z_2 = 1\left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right) = 0.309 + 0.951i$

$z_3 = 1\left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right) = -0.809 + 0.588i$

$z_4 = 1\left(\cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)\right) = -0.809 - 0.588i$

$z_5 = 1\left(\cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)\right) = 0.309 - 0.951i$

The roots of unity are 1, $0.309 + 0.951i$, $-0.809 + 0.588i$, $-0.809 - 0.588i$, $0.309 - 0.951i$, 1.



5. **Find the sixth roots of unity by using the method stated.**

   a. **Solve the polynomial equation $x^6 = 1$ algebraically.**

   $x^6 - 1 = 0$, $(x+1)(x^2 - x + 1)(x-1)(x^2 + x + 1) = 0$, $x = \pm 1$, $x = \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$, $x = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$

   The roots of unity are $1, -1, \frac{1}{2} + \frac{\sqrt{3}i}{2}, \frac{1}{2} - \frac{\sqrt{3}i}{2}, -\frac{1}{2} + \frac{\sqrt{3}i}{2}, -\frac{1}{2} - \frac{\sqrt{3}i}{2}$.

b. **Use the polar form $z^6 = r(\cos(\theta) + \sin(\theta))$, and find the modulus and argument of z.**

$z^6 = 1$, $r^6 = 1$, $r = 1$

$6\theta = 0$, $6\theta = 2\pi$, $6\theta = 4\pi$, $6\theta = 6\pi$, $6\theta = 8\pi$, $6\theta = 10\pi$, $6\theta = 12\pi$, $6\theta = 14\pi$, …; *therefore,* $\theta = 0, \frac{\pi}{3}, \frac{2}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi, \frac{7\pi}{3}, \ldots$

$z_1 = 1(\cos(0) + i\sin(0)) = 1$

$z_2 = 1\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

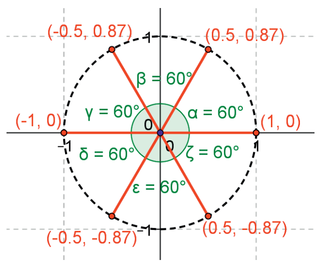
$z_3 = 1\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_4 = 1(\cos(\pi) + i\sin(\pi)) = -1$

$z_5 = 1\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$z_6 = 1\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

The roots of unity are $1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i$.



6. **Consider the equation $x^N = 1$ where N is a positive whole number.**

a. **For which value of N does $x^N = 1$ have only one solution?**

$N = 1$

b. **For which value of N does $x^N = 1$ have only ± 1 as solutions?**

$N = 2$

c. **For which value of N does $x^N = 1$ have only ± 1 and $\pm i$ as solutions?**

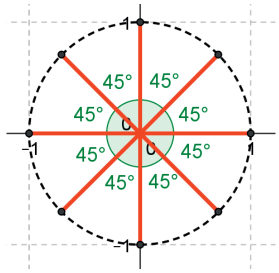
$N = 4$

d. **For which values of N does $x^N = 1$ have ± 1 as solutions?**

Any even number N produces solutions ± 1.

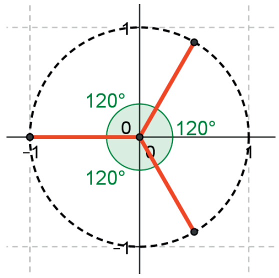
7. Find the equation that has the following solutions.

a.



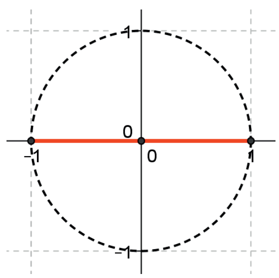
$x^8 = 1$

b.



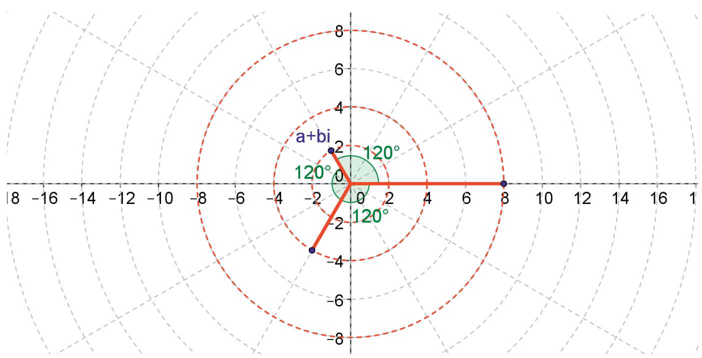
$x^3 = -1$

c.



$x^2 = 1$

8. Find the equation $(a + bi)^N = c$ that has solutions shown in the graph below.



$\left(-1 + \sqrt{3}i\right)^3 = 8$

Lesson 4: The Binomial Theorem

Student Outcomes

- Students discover patterns in the expansion of binomials, leading to the understanding of the binomial theorem.
- Students use Pascal's triangle to find the coefficients of binomial expansions.
- Students use binomial coefficients $C(n, k)$ to find the coefficients of binomial expansions.

Lesson Notes

Students begin the lesson by working through an exercise verifying that a given complex number is a solution to a given polynomial. By carrying out the tedious process of repeatedly multiplying binomial factors together, they should come to appreciate the usefulness of finding a quicker way to expand binomials raised to whole number powers.

Students generate Pascal's triangle recursively, and then the binomial coefficients $C(n, k) = \frac{n!}{k!(n-k)!}$ are introduced, and students connect the binomial coefficient $C(n, k)$ to the k^{th} element of row n of Pascal's triangle (counting the top row of the triangle as row 0). Students then connect the entries in row n of Pascal's triangle to the coefficients of the expansion of the binomial $(u + v)^n$. These connections are made explicit in the binomial theorem, which students apply to expand binomial expressions and to find specific terms in expansions (**A-APR.C.5**).

Consider splitting this lesson over two days, introducing Pascal's triangle and the binomial coefficients $C(n, k)$ on the first day and then connecting these to the binomial expansion $(u + v)^n$ and presenting the binomial theorem on the second day.

Classwork

Exercises 1–2 (4 minutes)

Assign half of the students to complete Exercise 1 and half of them to complete Exercise 2. Students should complete the exercise individually. After a few minutes, have them verify their responses with a partner assigned to the same exercise. Early finishers could write their solving process on chart paper to be displayed and explained when their classmates have completed the exercise. Review the exercises as an entire class. The purpose of this exercise is to show how tedious it can be to expand a binomial to higher powers so that students see the value in the formula presented in the binomial theorem.

Scaffolding:

- Show that $z = 1 + i$ is a solution to a simpler polynomial such as $z^2 - 2z + 2$.
- Challenge advanced learners to explain how properties of complex conjugates could be used to verify $1 - i$ is a solution if they know that $1 + i$ is a solution.

Exercises

1. **Show that $z = 1 + i$ is a solution to the fourth degree polynomial equation $z^4 - z^3 + 3z^2 - 4z + 6 = 0$.**

   If $(1+i)^4 - (1+i)^3 + 3(1+i)^2 - 4(1+i) + 6 = 0$, then $z = 1 + i$ is a solution.

   $(1+i)^2 = 1^2 + 2(i)(i) + i^2 = 2i$
   $(1+i)^3 = (1+i)(1+i)^2 = (1+i)(2i) = -2 + 2i$
   $(1+i)^4 = ((1+i)^2)^2 = (2i)^2 = 4i^2 = -4$

   $$\begin{aligned}(1+i)^4 - (1+i)^3 + 3(1+i)^2 - 4(1+i) + 6 &= -4 - (-2+2i) + 3(2i) - 4(1+i) + 6 \\ &= -4 + 2 - 2i + 6i - 4 - 4i + 6 \\ &= 0\end{aligned}$$

2. **Show that $z = 1 - i$ is a solution to the fourth degree polynomial equation $z^4 - z^3 + 3z^2 - 4z + 6 = 0$.**

   If $(1-i)^4 - (1-i)^3 + 3(1-i)^2 - 4(1-i) + 6 = 0$, then $z = 1 - i$ is a solution.

   $(1-i)^2 = 1^2 + 2(1)(-i) + (-i)^2 = -2i$
   $(1-i)^3 = (1-i)(1-i)^2 = (1-i)(-2i) = -2 - 2i$
   $(1-i)^4 = ((1-i)^2)^2 = (-2i)^2 = 4i^2 = -4$

   $$\begin{aligned}(1-i)^4 - (1-i)^3 + 3(1-i)^2 - 4(1-i) + 6 &= -4 - (-2-2i) + 3(-2i) - 4(1-i) + 6 \\ &= -4 + 2 + 2i - 6i - 4 + 4i + 6 \\ &= 0\end{aligned}$$

Discussion (8 minutes)

- What was most challenging or frustrating about verifying the solution in the previous exercises?
  - *It is tedious to carry out all the arithmetic needed to verify the solution.*
- How do you think these issues would be affected by the degree of the polynomial for which you are verifying a solution?
  - *The process becomes even more tedious and time consuming as the number of terms and degree of the polynomial increase.*
- Though it is tedious to substitute and simplify expressions to verify the solutions to polynomials, it is an important component to solving polynomials with complex solutions. What strategies did you use to try to expedite the process of simplifying in the exercise?
  - *Simplifying binomials of lesser degree and then applying those expressions to simplify binomials of higher degree helps to expedite the process of simplifying.*
- Instead of looking at specific complex numbers such as $1 + i$ or $1 - i$, let's look at any binomial expression $u + v$ where u and v can be numbers or expressions such as x, $2xy$, ab, etc. Let's suppose that we could write any expression $(u + v)^n$ in expanded form without having to multiply binomials repeatedly. We know how to do this for a few values of n. How can we write an expression equivalent to $(u + v)^0$ in expanded form? What about $(u + v)^1$?
  - $(u + v)^0 = 1$; $(u + v)^1 = u + v$

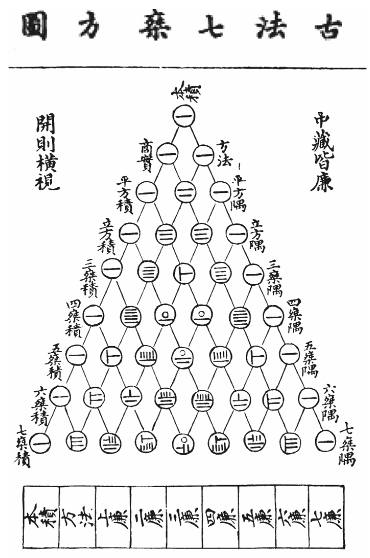
- You might also know an expanded expression to represent quadratic polynomials. What is the expanded form of the expression $(u+v)^2$?
  - $u^2 + 2uv + v^2$
- Our goal for this lesson is to find a quick way to expand polynomials of higher degree. We return to this task in a bit.
- Pascal's triangle is a triangular configuration of numbers that is constructed recursively.

$$\begin{array}{lccccccccccc}
\text{Row 0:} & & & & & & 1 & & & & & \\
\text{Row 1:} & & & & & 1 & & 1 & & & & \\
\text{Row 2:} & & & & 1 & & 2 & & 1 & & & \\
\text{Row 3:} & & & 1 & & 3 & & 3 & & 1 & & \\
\text{Row 4:} & & 1 & & 4 & & 6 & & 4 & & 1 & \\
\text{Row 5:} & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

- Rows in the triangle are generated recursively. The row at the top containing a single 1 is counted as Row 0. Row 1 contains two 1's. To build a row from the row above it, we start with a 1, written to the left of the position of the 1 from the previous row. In the next space, which is positioned horizontally between two elements in the row above, we add the elements in the upper row that are to the left and the right of the current position. That is, to generate Row 5 of the table, we start with a 1 and then add $1+4=5$, $4+6=10$, $6+4=10$, $4+1=5$ across the row and end with another 1.
- Now we want to generate Row 6 of Pascal's triangle. Allow students to suggest the entries as you record them in the triangle. Talk through the process of calculating each entry in Row 6: 1, then $1+5=6$ for the second entry, then $5+10=15$ for the third entry, etc.

$$\begin{array}{lccccccccccccc}
\text{Row 0:} & & & & & & & 1 & & & & & & \\
\text{Row 1:} & & & & & & 1 & & 1 & & & & & \\
\text{Row 2:} & & & & & 1 & & 2 & & 1 & & & & \\
\text{Row 3:} & & & & 1 & & 3 & & 3 & & 1 & & & \\
\text{Row 4:} & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
\text{Row 5:} & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
\text{Row 6:} & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

- Though its name comes from the French mathematician Blaise Pascal (1623–1662) who published the triangle in the Treatise on the Arithmetic Triangle in France in 1654, the use of the triangle predates Pascal. The figure on the following page was used as early as the thirteenth century and was known in China as Yang Hui's triangle. The markings in the circles on the following page are Chinese rod numbers, and they indicate the same numbers that we have in Pascal's triangle above.



"Yanghui Triangle" by Yáng Huī is licensed under CC BY 2.0
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- The Persian mathematician Omar Khayyam (1044–1123 C.E.) also mentioned the triangle in his works.

MP.7 & MP.8

- What patterns do you notice in the coefficients in the triangle? Think about this, and then share your ideas with a partner.
  - *Each row starts and ends with* 1*, and each of the rest of the coefficients is found by summing the two terms above it.*

Exercise 3 (2 minutes)

Students should complete the exercise individually and verify their responses with a partner when they are done. Have a student provide the coefficients for each row at the appropriate time.

3. Based on the patterns seen in Pascal's triangle, what would be the coefficients of Rows 7 and 8 in the triangle? Write the coefficients of the triangle beneath the part of the triangle shown.

Row 0:									**1**								
Row 1:								**1**		**1**							
Row 2:							**1**		**2**		**1**						
Row 3:						**1**		**3**		**3**		**1**					
Row 4:					**1**		**4**		**6**		**4**		**1**				
Row 5:				**1**		**5**		**10**		**10**		**5**		**1**			
Row 6:			**1**		**6**		**15**		**20**		**15**		**6**		**1**		
Row 7:		**1**		**7**		**21**		**35**		**35**		**21**		**7**		**1**	
Row 8:	**1**		**8**		**28**		**56**		**70**		**56**		**28**		**8**		**1**

Discussion (2 minutes)

- There is a way to calculate an entry of Pascal's triangle without writing out the whole triangle, but we first need to cover the idea of a *factorial*, which we denote by $n!$ for integers $n \geq 0$. First, we define $0! = 1$. Then, if $n > 0$, we define $n!$ to be the product of all positive integers less than or equal to n. For example, $2! = 2 \cdot 1$ and $3! = 3 \cdot 2 \cdot 1 = 6$. What are $4!$ and $5!$?
  - $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ *and* $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
- Then, for integers $n \geq 0$ and $k \geq n$, we define the quantity $C(n,k) = \frac{n!}{k!(n-k)!}$. For example, $C(6,4) = \frac{6!}{4! \cdot 2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{6 \cdot 5}{2 \cdot 1} = 15$.

Exercises 4–7 (6 minutes)

In these exercises, students practice calculating simple factorials and dividing factorials. Encourage students to write out the products and simplify the expression before calculating the factorials. Have students complete these exercises in pairs. Quickly debrief the answers before continuing, making sure that students understand that we can generate Row n of Pascal's triangle by calculating the quantities $C(n,k)$ for $0 \leq k \leq n$.

4. Calculate the following factorials.

a. $6!$

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

b. $10!$

$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3\,628\,800$

5. Calculate the value of the following factorial expressions.

a. $\frac{7!}{6!}$

$$\frac{7!}{6!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7$$

b. $\frac{10!}{6!}$

$$\frac{10!}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$$

c. $\frac{8!}{5!}$

$$\frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6 = 336$$

d. $\dfrac{12!}{10!}$

$$\frac{12!}{10!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 12 \cdot 11 = 132$$

6. **Calculate the following quantities.**

a. $C(1,0)$ and $C(1,1)$

$C(1,0) = \frac{1!}{0!1!} = 1$ *and* $C(1,1) = \frac{1!}{0!1!} = 1$

b. $C(2,0)$, $C(2,1)$, and $C(2,2)$

$C(2,0) = \frac{2!}{0!2!} = 1$, $C(2,1) = \frac{2!}{1!1!} = 2$, *and* $C(2,2) = \frac{2!}{2!0!} = 1$

c. $C(3,0)$, $C(3,1)$, $C(3,2)$, and $C(3,3)$

$C(3,0) = \frac{3!}{(0!3!)} = 1$, $C(3,1) = \frac{3!}{1!2!} = 3$, $C(3,2) = \frac{3!}{2!1!} = 3$, *and* $C(3,3) = \frac{3!}{3!0!} = 1$

d. $C(4,0)$, $C(4,1)$, $C(4,2)$, $C(4,3)$, and $C(4,4)$

$C(4,0) = \frac{4!}{0!4!} = 1$, $C(4,1) = \frac{4!}{1!3!} = 4$, $C(4,2) = \frac{4!}{2!2!} = 6$, $C(4,3) = \frac{4!}{3!1!} = 4$, *and* $C(4,4) = \frac{4!}{4!0!} = 1$

MP.7

7. **What patterns do you see in Exercise 6?**

The numbers $C(n,k)$ for $1 \leq n \leq 4$ give the same numbers as in Pascal's triangle.

Also, it appears that $C(n,0) = 1$ and $C(n,n) = 1$ for each n.

Exercises 8–11 (7 minutes)

We now return to looking for a shortcut to expanding a binomial expression. In these exercises, students connect the binomial coefficients $C(n,k)$ to the coefficients of the binomial expansion $(u+v)^n$.

Have the students complete these exercises in pairs. At an appropriate time, have students display their solutions.

8. **Expand the expression $(u+v)^3$.**

$$\begin{aligned}(u+v)^3 &= (u+v)(u^2+2uv+v^2)\\ &= u^3+2u^2v+uv^2+vu^2+2uv^2+v^3\\ &= u^3+3u^2v+3uv^2+v^3\end{aligned}$$

9. **Expand the expression $(u+v)^4$.**

$$\begin{aligned}(u+v)^4 &= (u+v)(u^3+3u^2v+3uv^2+v^3)\\ &= u^4+3u^3v+3u^2v^2+uv^3+vu^3+3u^2v^2+3uv^3+v^4\\ &= u^4+4u^3v+6u^2v^2+4uv^3+v^4\end{aligned}$$

Scaffolding:

Encourage students to write each term in the expansion with the power of u preceding the power of v if they are struggling to recognize like terms.

10.

a. **Multiply the expression you wrote in Exercise 9 by u.**

$u(u^4+4u^3v+6u^2v^2+4uv^3+v^4)=u^5+4u^4v+6u^3v^2+4u^2v^3+uv^4$

b. **Multiply the expression you wrote in Exercise 9 by v.**

$v(u^4+4u^3v+6u^2v^2+4uv^3+v^4)=u^4v+4u^3v^2+6u^2v^3+4uv^4+v^5$

c. **How can you use the results from parts (a) and (b) to find the expanded form of the expression $(u+v)^5$?**

Because $(u+v)^5=(u+v)(u+v)^4=u(u+v)^4+v(u+v)^4$, we have

$$\begin{aligned}(u+v)^5 &= (u^5+4u^4v+6u^3v^2+4u^2v^3+uv^4)+(u^4v+4u^3v^2+6u^2v^3+4uv^4+v^5)\\ &= u^5+5u^4v+10u^3v^2+10u^2v^3+5uv^4+v^5.\end{aligned}$$

11. **What do you notice about your expansions for $(u+v)^4$ and $(u+v)^5$? Does your observation hold for other powers of $(u+v)$?**

The coefficients of $(u+v)^4$ are the numbers in Row 4 of Pascal's triangle. The coefficients of $(u+v)^5$ are the numbers in Row 5 of Pascal's triangle. The same pattern holds for $(u+v)$, $(u+v)^2$, and $(u+v)^3$.

Discussion (5 minutes)

This teacher-led discussion reiterates the connection that students made in the previous exercises between the coefficients of a binomial expansion and Pascal's triangle.

- So now we have computed the expansions for $(u+v)^n$ starting with $n=0$ to $n=5$. Let's arrange them vertically, from least power to the greatest and with all the coefficients written explicitly.

$$\begin{aligned}(u+v)^0 &= 1\\ (u+v)^1 &= 1u+1v\\ (u+v)^2 &= 1u^2+2uv+1v^2\\ (u+v)^3 &= 1u^3+3u^2v+3uv^2+1v^3\\ (u+v)^4 &= 1u^4+4u^3v+6u^2v^2+4uv^3+1v^4\\ (u+v)^5 &= 1u^5+5u^4v+10u^3v^2+10u^2v^3+5uv^4+1v^5\end{aligned}$$

MP.7 & MP.8

- What patterns do you notice in the expansion? Think about this for a minute, and then share your ideas with a partner.
  - *Within each row, the power of u decreases from left to right, and the power of v increases from left to right; the sum of the powers in each term is equal to the power of the binomial; each row begins and ends with terms that have a coefficient of 1; the number of terms is one greater than the power of the binomial.*
  - *The coefficients of the binomial expansion are the numbers in the corresponding row of Pascal's triangle.*
  - *The coefficients of the binomial expansion $(u+v)^n$ are $C(n,k)$ as k increases from 0 to n.*

MP.7 & MP.8

- The correspondence between the numbers in Row n of Pascal's triangle and the coefficients of the expanded expression $(u+v)^n$ is known as *the binomial theorem*. The numbers $C(n,k)$ are called *binomial coefficients*.
- **THE BINOMIAL THEOREM:** For any expressions u and v,
  $(u+v)^n = u^n + C(n,1)u^{n-1}v + C(n,2)u^{n-2}v^2 + \cdots + C(n,k)u^{n-k}v^k + \cdots + C(n,n-1)u\,v^{n-1} + v^n$.
  That is, the coefficients of the expanded binomial $(u+v)^n$ are exactly the numbers in Row n of Pascal's triangle.

Exercise 12 (4 minutes)

Have students work on these exercises in pairs or small groups. If time permits, have students share their results with the class either on the document camera, using individual white boards, or by writing on the board.

12. Use the binomial theorem to expand the following binomial expressions.

a. $(x+y)^6$

$\mathbf{x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6}$

b. $(x+2y)^3$

$\mathbf{x^3 + 6x^2y + 12xy^2 + 8y^3}$

c. $(ab+bc)^4$

$\mathbf{a^4b^4 + 4a^3b^4c + 6a^2b^4c^2 + 4ab^4c^3 + b^4c^4}$

d. $(3xy-2z)^3$

$\mathbf{27x^3y^3 - 54x^2y^2z + 36xyz^2 - 8z^3}$

e. $(4p^2qr - qr^2)^5$

$\mathbf{1024p^{10}q^5r^5 - 1280p^8q^5r^6 + 640p^6q^5r^7 - 160p^4q^5r^8 + 20p^2q^5r^9 - q^5r^{10}}$

Closing (3 minutes)

Have the students reflect on the questions. After a minute, have them share their responses with a partner. If time permits, a few students could share their reflections with the rest of the class.

- When is it helpful to apply the binomial theorem?
  - *The binomial theorem can be used to expand binomials in the form $(u+v)^n$ without having to multiply several factors together.*
- How is Pascal's triangle helpful when applying the binomial theorem?
  - *The entries in Row n represent the coefficients of the expansion of $(u+v)^n$.*

- How are the binomial coefficients $C(n, k)$ helpful when applying the binomial theorem?
  - *The binomial coefficients allow us to calculate Row n of Pascal's triangle without writing out all of the previous rows.*

Lesson Summary

Pascal's triangle is an arrangement of numbers generated recursively:

Row 0:						**1**					
Row 1:					**1**		**1**				
Row 2:				**1**		**2**		**1**			
Row 3:			**1**		**3**		**3**		**1**		
Row 4:		**1**		**4**		**6**		**4**		**1**	
Row 5:	**1**		**5**		**10**		**10**		**5**		**1**
	⋮		⋮		⋮		⋮		⋮		⋮

For an integer $n \geq 1$, the number $n!$ is the product of all positive integers less than or equal to n. We define $0! = 1$.

The binomial coefficients $C(n, k)$ are given by $C(n, k) = \frac{n!}{k!(n-k)!}$ for integers $n \geq 0$ and $0 \leq k \leq n$.

THE BINOMIAL THEOREM: For any expressions u and v,

$$(u + v)^n = u^n + C(n, 1)u^{n-1}v + C(n, 2)u^{n-2}v^2 + \cdots + C(n, k)u^{n-k}v^k + \cdots + C(n, n-1)u\,v^{n-1} + v^n.$$

That is, the coefficients of the expanded binomial $(u + v)^n$ are exactly the numbers in Row n of Pascal's triangle.

Exit Ticket (4 minutes)

Name ______________________________ Date ______________

Lesson 4: The Binomial Theorem

Exit Ticket

1. Evaluate the following expressions.
   a. $5!$

   b. $\frac{8!}{6!}$

   c. $C(7,3)$

2. Find the coefficients of the terms below in the expansion of $(u+v)^8$. Explain your reasoning.
   a. u^2v^6

   b. u^3v^5

   c. u^4v^4

Exit Ticket Sample Solutions

1. **Evaluate the following expressions.**

   a. $5!$

   $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

   b. $\frac{8!}{6!}$

   $\frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56$

   c. $C(7,3)$

   $C(7,3) = \frac{7!}{3!\,4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(4 \cdot 3 \cdot 2 \cdot 1)} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

   Alternatively, students could use the corresponding entry of Row 7 of Pascal's triangle, $7\ 21\ 35\ 35\ 21\ 7\ 1$, *which is* 35.

2. **Find the coefficients of the terms below in the expansion of $(u+v)^8$. Explain your reasoning.**

   a. u^2v^6

   The binomial theorem says that the u^2v^6 term of the expansion is $C(8,2)u^2v^6$, so the coefficient is $C(8,2) = \frac{8!}{2!6!} = 28$. Alternatively, Row 8 of Pascal's triangle is $1\ 8\ 28\ 56\ 70\ 56\ 28\ 8\ 1$, and the entry corresponding to u^2v^6 is 28.

   b. u^3v^5

   The binomial theorem says that the u^3v^5 term of the expansion is $C(8,3)u^3v^5$, so the coefficient is $C(8,3) = \frac{8!}{3!5!} = 56$. Alternatively, Row 8 of Pascal's triangle is $1\ 8\ 28\ 56\ 70\ 56\ 28\ 8\ 1$, and the entry corresponding to u^3v^5 is 56.

   c. u^4v^4

   The binomial theorem says that the u^4v^4 term of the expansion is $C(8,4)u^4v^4$, so the coefficient is $C(8,4) = \frac{8!}{4!4!} = 70$. Alternatively, Row 8 of Pascal's triangle is $1\ 8\ 28\ 56\ 70\ 56\ 28\ 8\ 1$, and the entry corresponding to u^4v^4 is 70.

Problem Set Sample Solutions

1. **Evaluate the following expressions.**

   a. $\frac{9!}{8!}$

   $$\frac{9!}{8!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 9$$

   b. $\frac{7!}{5!}$

   $$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 6 = 42$$

   c. $\frac{21!}{19!}$

   $$\frac{21!}{19!} = \frac{21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot \dots \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{19 \cdot 18 \cdot 17 \cdot \dots \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 21 \cdot 20 = 420$$

   d. $\frac{8!}{4!}$

   $$\frac{8!}{4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$$

2. **Use the binomial theorem to expand the following binomial expressions.**

   a. $(x+y)^4$

   $$\begin{aligned}(x+y)^4 &= x^4 + C(4,1)x^3y + C(4,2)x^2y^2 + C(4,3)xy^3 + y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

   b. $(x+2y)^4$

   $$\begin{aligned}(x+2y)^4 &= x^4 + C(4,1)x^3(2y) + C(4,2)x^2(2y)^2 + C(4,3)x(2y)^3 + (2y)^4 \\ &= x^4 + 4x^3(2y) + 6x^2(4y^2) + 4x(8y^3) + 16y^4 \\ &= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4\end{aligned}$$

   c. $(x+2xy)^4$

   $$\begin{aligned}(x+2xy)^4 &= x^4 + C(4,1)x^3(2xy) + C(4,2)x^2(2xy)^2 + C(4,3)x(2xy)^3 + (2xy)^4 \\ &= x^4 + 4x^3(2xy) + 6x^2(4x^2y^2) + 4x(8x^3y^3) + 16x^4y^4 \\ &= x^4 + 8x^4y + 24x^4y^2 + 32x^4y^3 + 16x^4y^4\end{aligned}$$

   d. $(x-y)^4$

   $$\begin{aligned}(x-y)^4 &= \big(x+(-y)\big)^4 \\ &= x^4 + C(4,1)x^3(-y) + C(4,2)x^2(-y)^2 + C(4,3)x(-y)^3 + (-y)^4 \\ &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4\end{aligned}$$

e. $(x - 2xy)^4$

$$(x - 2xy)^4 = x^4 + C(4,1)x^3(-2xy) + C(4,2)x^2(-2xy)^2 + C(4,3)x(-2xy)^3 + (-2xy)^4$$
$$= x^4 + 4x^3(-2xy) + 6x^2(4x^2y^2) + 4x(-8x^3y^3) + 16x^4y^4$$
$$= x^4 - 8x^4y + 24x^4y^2 - 32x^4y^3 + 16x^4y^4$$

3. Use the binomial theorem to expand the following binomial expressions.

a. $\left(1+\sqrt{2}\right)^5$

$$\left(1+\sqrt{2}\right)^5 = 1^5 + C(5,1)1^4\sqrt{2} + C(5,2)1^3\left(\sqrt{2}\right)^2 + C(5,3)1^2\left(\sqrt{2}\right)^3 + C(5,4)1\cdot\left(\sqrt{2}\right)^4 + \left(\sqrt{2}\right)^5$$
$$= 1 + 5\sqrt{2} + 10\cdot 2 + 10\cdot 2\sqrt{2} + 5\cdot 4 + 4\sqrt{2}$$
$$= 41 + 29\sqrt{2}$$

b. $(1+i)^9$

$$(1+i)^9 = 1 + 9i + 36i^2 + 84i^3 + 126i^4 + 126i^5 + 84i^6 + 36i^7 + 9i^8 + i^9$$
$$= 1 + 9i - 36 - 84i + 126 + 126i - 84 - 36i + 9 + i$$
$$= 16 + 16i$$

c. $(1-\pi)^5$ (Hint: $1 - \pi = 1 + (-\pi)$.)

$$(1-\pi)^5 = 1 + C(5,1)(-\pi) + C(5,2)(-\pi)^2 + C(5,3)(-\pi)^3 + C(5,4)(-\pi)^4 + (-\pi)^5$$
$$= 1 - 5\pi + 10\pi^2 - 10\pi^3 + 5\pi^4 - \pi^5$$

d. $\left(\sqrt{2}+i\right)^6$

$$\left(\sqrt{2}+i\right)^6 = \left(\sqrt{2}\right)^6 + C(6,1)\left(\sqrt{2}\right)^5 i + C(6,2)\left(\sqrt{2}\right)^4 i^2 + C(6,3)\left(\sqrt{2}\right)^3 i^3 + C(6,4)\left(\sqrt{2}\right)^2 i^4 + C(6,5)\sqrt{2}i^5 + i^6$$
$$= 8 + 6\cdot 4\sqrt{2}i + 15\cdot 4(-1) + 20\cdot 2\sqrt{2}(-i) + 15\cdot 2\cdot 1 + 6\sqrt{2}(i) + (-1)$$
$$= -23 - 10\sqrt{2}i$$

e. $(2-i)^6$

$$(2-i)^6 = \left(2+(-i)\right)^6$$
$$= 2^6 + C(6,1)2^5(-i) + C(6,2)2^4(-i)^2 + C(6,3)2^3(-i)^3 + C(6,4)2^2(-i)^4 + C(6,5)2(-i)^5 + (-i)^6$$
$$= 64 - 6\cdot 32i + 15\cdot 16(-1) + 20\cdot 8(i) + 15\cdot 4\cdot 1 + 6\cdot 2(-i) - 1$$
$$= -117 - 44i$$

4. Consider the expansion of $(a+b)^{12}$. Determine the coefficients for the terms with the powers of a and b shown.

a. a^2b^{10}

$$(a+b)^{12} = a^{12} + C(12,1)a^{11}b + \cdots + C(12,10)a^2b^{10} + C(12,11)ab^{11} + b^{12}$$

So, the coefficient of a^2b^{10} is $C(12,10) = \frac{12!}{2!10!} = \frac{12\cdot 11}{2\cdot 1} = 66$.

b. a^5b^7

The coefficient of a^5b^7 is $C(12,7) = \frac{12!}{5!7!} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 792$.

c. a^8b^4

The coefficient of a^8b^4 is $C(12,4) = \frac{12!}{8!4!} = \frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1} = 495$.

5. Consider the expansion of $(x+2y)^{10}$. Determine the coefficients for the terms with the powers of x and y shown.

a. x^2y^8

The x^2y^8 term is $C(10,8)x^2(2y)^8 = 45x^2 \cdot 256y^8 = 11520\, x^2y^8$, so the coefficient of x^2y^8 is $11,520$.

b. x^4y^6

The x^4y^6 term is $C(10,6)x^4(2y)^6 = 210x^4 \cdot 64y^6 = 13440\, x^4y^6$, so the coefficient of x^4y^6 is $13,440$.

c. x^5y^5

The x^5y^5 term is $C(10,5)x^5(2y)^5 = 252x^2 \cdot 32y^8 = 8064\, x^5y^5$, so the coefficient of x^5y^5 is $8,064$.

6. Consider the expansion of $(5p+2q)^6$. Determine the coefficients for the terms with the powers of p and q shown.

a. p^2q^4

Since $C(6,2) = 15$ and $15(5p)^2(2q)^4 = 15(25p^2)(16q^4) = 6000p^2q^4$, the coefficient is $6,000$.

b. p^5q

Since $C(6,5) = 6$ and $6(5p)^5(2q) = 6(3125p^5)(2q) = 37500p^5q$, the coefficient is $37,500$.

c. p^3q^3

Since $C(6,3) = 20$ and $20(5p)^3(2q)^3 = 20(125p^3)(8q^3) = 20000p^3q^3$, the coefficient is $20,000$.

7. Explain why the coefficient of the term that contains u^n is 1 in the expansion of $(u+v)^n$.

The corresponding binomial coefficient is $C(n,0) = \frac{n!}{0!n!} = \frac{1}{0!} = 1$.

8. Explain why the coefficient of the term that contains $u^{n-1}v$ is n in the expansion of $(u+v)^n$.

The corresponding binomial coefficient is $C(n,1) = \frac{n!}{1!(n-1)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1}{(1)\big((n-1)\cdot(n-2)\cdots 2\cdot 1\big)} = n$.

9. Explain why the rows of Pascal's triangle are symmetric. That is, explain why $C(n,k) = C\big(n,(n-k)\big)$.

Using the formula for the binomial coefficients, $C(n,k) = \frac{n!}{k!(n-k)!}$ and $C(n,n-k) = \frac{n!}{(n-k)!(n-(n-k)!)}$, so $C(n,n-k) = \frac{n!}{(n-k)!n!} = C(n,k)$.

Lesson 5: The Binomial Theorem

Student Outcomes

- Students observe patterns in the coefficients of the terms in binomial expansions. They formalize their observations and explore the mathematical basis for them.
- Students use the binomial theorem to solve problems in a geometric context.

Lesson Notes

This lesson provides students with opportunities to explore additional patterns formed by the coefficients of binomial expansions. They apply the binomial theorem to find a mathematical basis for the patterns observed. They also apply the theorem to explore average rates of change for the volume of a sphere with a changing radius.

Classwork

Opening Exercise (3 minutes)

Students should complete the problems independently or in pairs. They could write the solutions on paper or display them on small white boards for quick checks. Remind students that the top row in Pascal's triangle is Row 0.

MP.7

Opening Exercise

Write the first six rows of Pascal's triangle. Then, use the triangle to find the coefficients of the terms with the powers of u and v shown, assuming that all expansions are in the form $(u+v)^n$. Explain how Pascal's triangle allows you to determine the coefficient.

$$\begin{array}{ccccccccccccc} &&&&&&1&&&&&& \\ &&&&&1&&1&&&&& \\ &&&&1&&2&&1&&&& \\ &&&1&&3&&3&&1&&& \\ &&1&&4&&6&&4&&1&& \\ &1&&5&&10&&10&&5&&1& \\ 1&&6&&15&&20&&15&&6&&1 \end{array}$$

a. u^2v^4

15; this is in Row 6 (sum of exponents is 6) and the fifth term since the power of u started at 6 and has decreased to 2, which takes 5 steps.

b. u^3v^2

10; this is the third term in Row 5.

Scaffolding:

- For students struggling with the Opening Exercise, show an "amended" version of the first three rows of Pascal's triangle that includes the associated power of $(u+v)^n$:

  0 1
  1 1 1
  2 1 2 1
  3 1 3 3 1

  Ask questions such as, "What patterns do you see?" and "What does the number in the leftmost column represent?" Expand $(u+v)^0$, $(u+v)^1$, $(u+v)^2$, and describe how the results are related to Pascal's triangle.
- Show $(u+v)^2 = u^2 + 2uv + v^2$ and $(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$. Ask, "What patterns do you notice in the powers of u and v?"

c. u^2v^2

6; this is the third term in Row 4.

d. v^{10}

1; this is the last term in Row 10 since there is no u.

Discussion (3 minutes)

- In the previous exercise, how did you determine which row of Pascal's triangle to use to find the requested coefficient? Provide an example.
  - *The sum of the powers of u and v equals n. This is the row of Pascal's triangle that contains the appropriate coefficients. For example, I used Row 6 of Pascal's triangle to find the coefficient of the term that contains u^2v^4 because the sum of the powers of u and v is 6.*
- Once you found the appropriate row of Pascal's triangle, how did you determine which coefficient to use?
  - *The power of v is the same as the term in the expansion (e.g., the coefficient for u^2v^4 is the fourth term after the initial 1 in the sixth row).*
- And how could you use Pascal's triangle to find the coefficient of v^{10} in part (d) without expanding the triangle to Row 10?
  - *This is the last term in the expansion of $(u+v)^{10}$, and the last coefficient in any binomial expansion is 1.*
- Good. And how could you use Pascal's triangle to find the coefficient of uv^6 in part (e) without writing out the seventh row?
  - *The coefficient of the uv^{n-1} term in the expansion of $(u+v)^n$ is n. The coefficient can be found by adding the coefficients of the last two numbers in Row 6 of the triangle.*

Example 1 (7 minutes)

The example should be completed as a teacher-led discussion. It builds upon initial patterns in binomial expansions explored in Lesson 4. Students recognize a pattern in the alternating sums of each row of Pascal's triangle and see how the binomial theorem can be used to provide a mathematical explanation for the observed pattern.

- Let's look again at the first six rows of Pascal's triangle. What patterns do you remember in the coefficients of the terms within each row?
  - *Each row begins and ends with 1, and the rows are symmetric (e.g., the value of the first term is the same as the $(n-1)^{st}$ term, the second term is the same as the $(n-2)^{nd}$ term).*
- And what is the relationship of the values between rows?
  - *If we disregard the 1's, the value of any entry in the triangle is the sum of the two values immediately above it.*

- So, if we were going to write out Row 7 for the triangle, how could we find the entries?
  - *The row would start and end with* 1*, and the rest of the terms would be found by adding adjacent terms from Row 6.*
- What would this look like?
  - $1\ (1+6)\ (6+15)\ (15+20)\ (20+15)\ (15+6)\ (6+1)\ 1 \leftrightarrow 1\ 7\ 21\ 35\ 35\ 21\ 7\ 1$
- Good. Let's look at another interesting pattern in Pascal's triangle. First, let's compute the alternating sums of the rows of the triangle. For an alternating sum, we alternate adding and subtracting numbers. For Row 1 of the triangle, which has entries 1 and 1, the alternating sum is $1-1=0$. For Row 2 of the triangle, which has entries $1\ 2\ 1$, the alternating sum would be $1-2+1=0$. What is the value of the alternating sum of Row 3?
  - $1-3+3-1=0$
- Good. Now if we look at the alternating sums, what pattern do you see?
  - With the exception of the top row, the alternating sums are all 0.
- Why do you think the sums are all 0? Share your ideas with a partner.

Answers will vary. Encourage the students to form conjectures and share them (e.g., perhaps the symmetry of the triangle affects the alternating sums).

- Let's try to use the binomial theorem to explore this pattern by rewriting 0 as the sum $1+(-1)$. Now what is the value of $\left(1+(-1)\right)^n$? How do you know?
  - It is 0 because the expression $1+(-1)$ is equal to 0 and $0^n=0$.
- So, $0=\left(1+(-1)\right)^n$. Now let's use the binomial theorem to expand $\left(1+(-1)\right)^n$. How can we apply the binomial theorem to this expansion?
  - Substitute $u=1$ and $v=-1$:

$$(u+v)^n = u^n + A_1u^{n-1}v + A_2u^{n-2}v^2 + \cdots + A_{n-1}uv^{n-1} + v^n$$

$$0 = 1^n + A_1(1^{n-1})(-1) + A_2(1^{n-2})(-1)^2 + \cdots + A_{n-1}(1)(-1)^{n-1} + (-1)^n$$

- How else can we write this equation?
  - $0 = 1 - A_1 + A_2 - \cdots + A_{n-1}(1)(-1)^{n-1} + (-1)^n$
- Why did we not simplify the last two terms in the expansion?
  - *The signs vary based on the value of* n*. If* n *is even, the last terms are* $-A_{n-1}$ *and* 1*. If* n *is odd, the last terms are* A_{n-1} *and* -1*.*
- In both cases, what does our expansion represent?
  - *The coefficients of the* n^{th} *row of Pascal's triangle with alternating signs*
- And what can we conclude from this?
  - *The alternating sum of the* n^{th} *row of Pascal's triangle is always* 0*.*

Example 1

Look at the alternating sums of the rows of Pascal's triangle. An alternating sum alternately subtracts and then adds values. For example, the alternating sum of Row 2 would be $1 - 2 + 1$, and the alternating sum of Row 3 would be $1 - 3 + 3 - 1$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

a. **Compute the alternating sum for each row of the triangle shown.**

The sums are all zero.

b. **Use the binomial theorem to explain why each alternating sum of a row in Pascal's triangle is 0.**

The binomial theorem states that $(u + v)^n = u^n + A_1u^{n-1}v + A_2u^{n-2}v^2 + A_{n-1}uv^{n-1} + n$.

So, $0 = \left(1 + (-1)\right)^n = 1^n + A_1(1^{n-1})(-1) + A_2(1^{n-2})(-1)^2 + \cdots + A_{n-1}(1)(-1)^{n-1} + (-1)^n = 1 - A_1 + A_2 + \cdots - A_{n-1} + 1$ for all even values of n or $1 - A_1 + A_2 + \cdots + A_{n-1} - 1$ for all odd values of n.

Exercises 1–2 (15 minutes)

The students should be placed into small groups. Assign the groups to complete either Exercise 1 or Exercise 2. After about 5 minutes, the groups assigned to the same exercise could work together to discuss their findings and to organize their thoughts in order to prepare to present the exercise to the rest of the class. During the final 5 minutes, students present their solutions to the class.

Scaffolding:

- Students in groups that are struggling with how to approach Exercise 1 could be prompted to write 2^n as $(1 + 1)^n$.
- Students in groups that are struggling with how to approach Exercise 2 could be prompted to write 11^n as$(10 + 1)^n$.
- Help struggling students convert from the Pascal's triangle format to base-ten format of 11^n by writing $1|5|10|10|5|1$ in expanded form (i.e., $1 \times (100\,000) + 5 \times (10000) + 10 \times (1000) + 10 \times (100) + 5 \times (10) + 1 = 161\,051$).
- Advanced students could explore a different pattern they covered related to Pascal's triangle.

Exercises 1–2

1. **Consider Rows 0–6 of Pascal's triangle.**

a. **Find the sum of each row.**

$$
\begin{array}{cccccccccccccc}
 & & & & & & & & & & & & & \textit{SUM} \\
 & & & & & & 1 & & & & & & & 1 \\
 & & & & & 1 & & 1 & & & & & & 2 \\
 & & & & 1 & & 2 & & 1 & & & & & 4 \\
 & & & 1 & & 3 & & 3 & & 1 & & & & 8 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & & 16 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & 32 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & 64
\end{array}
$$

b. **What pattern do you notice in the sums computed?**

The sum of the coefficients in the n^{th} row appears to be 2^n.

c. **Use the binomial theorem to explain this pattern.**

$2^n = (1+1)^n$

The binomial theorem states that $(u+v)^n = u^n + A_1 u^{n-1} v + A_2 u^{n-2} v^2 + \cdots + A_{n-1} u v^{n-1} + v^n$.

So,
$(1+1)^n = 1^n + A_1(1^{n-1})(1) + A_2(1^{n-2})(1)^2 + \cdots + A_{n-1}(1)(1)^{n-1} + 1^n = 1 + A_1 + A_2 + \cdots + A_{n-1} + 1$,
which is the sum of the coefficients of the n^{th} *row of Pascal's triangle.*

2. **Consider the expression** 11^n.

a. **Calculate** 11^n, **where** $n = 0, 1, 2, 3, 4$.

$11^0 = 1$
$11^1 = 11$
$11^2 = 121$
$11^3 = 1331$
$11^4 = 14641$

b. **What pattern do you notice in the successive powers?**

The digits of 11^n *correspond to the coefficients in the* n^{th} *row of Pascal's triangle.*

MP.2 & MP.7

c. **Use the binomial theorem to demonstrate why this pattern arises.**

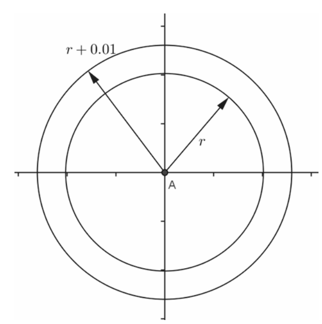
$$11^n = (10+1)^n = 10^n + A_1(10)^{n-1}(1) + A_2(10)^{n-2}(1)^2 + \cdots + A_{n-1}(10)(1)^{n-1} + 1^n$$
$$= 10^n + A_1(10)^{n-1} + A_2(10)^{n-2} + \cdots + A_{n-1}(10) + 1$$

d. **Use a calculator to find the value of** 11^5. **Explain whether this value represents what would be expected based on the pattern seen in lower powers of** 11.

If we continued the pattern seen in 11^n, *where* $n = 0, 1, 2, 3, 4$, *we would expect* 11^5 *to comprise the digits of the fifth row in Pascal's triangle. In other words, we could conjecture that* $11^5 = 1|5|10|10|5|1$. *Because we cannot represent a* 10 *as a single digit, the number on the calculator would be* $161{,}051$.

Scaffolding:

Students who are strong visual learners may benefit from being shown a model of the situation (e.g., a solid sphere with radius $r + 0.01$ units with a sphere with radius r removed). Think of it as the shell of a tennis ball with this cross-section.



Example 2 (7 minutes)

This example should be completed as a teacher-led discussion. It provides an opportunity for the students to apply the binomial theorem to a geometric context.

- We have explored patterns using the binomial theorem. Let's see how it can also be applied to solving problems with geometric solids. In Example 2, how can we calculate the increase in volume from a sphere with radius r to radius $r + 0.01$ units?
  - *Subtract the volume of the larger sphere from that of the smaller sphere.*
- How can we represent this mathematically?
  - $V(r+0.01) - V(r) = \frac{4}{3}\pi(r+0.01)^3 - \frac{4}{3}\pi r^3$
- How can we use the binomial theorem to simplify this expression?
  - *We can use it to expand* $\frac{4}{3}\pi(r+0.01)^3$, *where* $u = r$, $v = 0.01$, *and* $n = 3$.

- Once we expand the expression and combine like terms, we are left with $0.04\pi(r)^2 + 0.0004\pi r + 0.000\,001\pi$. How can we use this expression to find the average rate of change of the volume?
  - *Divide the expression by the change in the radius, which is* 0.01.
- This gives us $4\pi r^2 + 0.04\pi r + 0.0001\pi$. How can we say that this approximates the surface area $S(r)$ when there are three terms in our expression?
  - *For most values of r, the values 0.04π and 0.0001π are negligible in comparison to the value of $4\pi r^2$, so we can reasonably approximate the expression as $4\pi r^2$, which is the surface area $S(r)$.*
- Why does the expression $V(r + 0.01) - V(r)$ represent a shell with thickness 0.01 units covering the outer surface of the sphere with radius r?
  - *It is the surface that results from starting with a solid sphere with radius $r + 0.01$ and removing from it the solid sphere with radius r.*
- Why can we approximate the volume of the shell as $0.01 \cdot S(r)$?
  - *If we were to "unroll" the shell so the surface area of its inside lay flat, the expression $0.01 \cdot S(r)$ would be equal to the area of the flat surface multiplied by its height.*
- And using this expression for volume, how could we calculate the average rate of change?
  - *Divide by 0.01, which results in $S(r)$.*

Example 2

We know that the volume $V(r)$ and surface area $S(r)$ of a sphere of radius r are given by these formulas:

$$V(r) = \frac{4}{3}\pi r^3$$

$$S(r) = 4\pi r^2$$

Suppose we increase the radius of a sphere by 0.01 units from r to $r + 0.01$.

a. Use the binomial theorem to write an expression for the increase in volume.

$$\begin{aligned} V(r+0.01) - V(r) &= \frac{4}{3}\pi(r+0.01)^3 - \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi\left((r)^3 + 3(r)^2(0.01) + 3r(0.01)^2 + (0.01)^3\right) - \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi r^3 + 0.04\pi(r)^2 + 0.0004\pi r + 0.000\,001\pi - \frac{4}{3}\pi r^3 \\ &= 0.04\pi(r)^2 + 0.0004\pi r + 0.000\,001\pi \end{aligned}$$

b. Write an expression for the average rate of change of the volume as the radius increases from r to $r + 0.01$.

$$\text{Average rate of change} = \frac{V(r+0.01) - V(r)}{(r+0.01) - r}$$

c. Simplify the expression in part (b) to compute the average rate of change of the volume of a sphere as the radius increases from r to $r + 0.01$.

$$\text{Average rate of change} = \frac{0.04\pi(r)^2 + 0.0004\pi r + 0.000\,001\pi}{0.01} = 4\pi r^2 + 0.04\pi r + 0.0001\pi$$

d. **What does the expression from part (c) resemble?**

Surface area of the sphere with radius r

e. **Why does it make sense that the average rate of change should approximate the surface area? Think about the geometric figure formed by $V(r+0.01)-V(r)$. What does this represent?**

It is a shell of volume, a layer 0.01 units thick, covering the surface area of the inner sphere of radius r.

f. **How could we approximate the volume of the shell using surface area? And the average rate of change for the volume?**

The volume is approximately $S(r)\cdot(0.01)$; rate of change of volume is approximately $\frac{S(r)\times 0.01}{0.01}=S(r)$, which is the surface area of the sphere, $S(r)$.

Closing (5 minutes)

Have the students respond in writing to the prompt. After a few minutes, select several students to share their responses.

- Why is it beneficial to understand and be able to apply the binomial theorem? After you respond, share your thoughts with a partner.
  - *The binomial theorem can help determine whether complex numbers are solutions to polynomial functions.*
  - *The binomial theorem can explain mathematical patterns of numbers and patterns seen in Pascal's triangle.*
  - *The binomial theorem can be used to solve problems involving geometric solids.*
  - *The binomial theorem expedites the process of expanding binomials raised to whole number powers greater than 1.*

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 5: The Binomial Theorem

Exit Ticket

The area and circumference of a circle of radius r are given by

$$A(r) = \pi r^2$$

$$C(r) = 2\pi r$$

a. Show mathematically that the average rate of change of the area of the circle as the radius increases from r to $r + 0.01$ units is very close to the circumference of the circle.

b. Explain why this makes sense geometrically.

Exit Ticket Sample Solutions

The area and circumference of a circle of radius r are given by

$$A(r) = \pi r^2$$
$$C(r) = 2\pi r$$

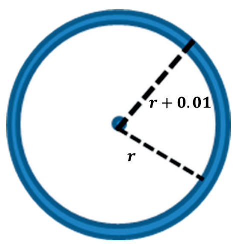
a. Show mathematically that the average rate of change of the area of the circle as the radius increases from r to $r + 0.01$ units is very close to the circumference of the circle.

Average rate of change: $\dfrac{A(r+0.01) - A(r)}{(r+0.01) - r}$

$$\begin{aligned} A(r+0.01) - A(r) &= \pi(r+0.01)^2 - \pi r^2 \\ &= \pi r^2 + 2\pi(r)(0.01) + \pi(0.01)^2 - \pi r^2 \\ &= 0.02\pi r + 0.0001\pi \end{aligned}$$

Average rate of change: $\dfrac{0.02\pi r + 0.0001\pi}{0.01} = 2\pi r + 0.01\pi$*, which is approximately equal to* $C(r)$*.*

b. Explain why this makes sense geometrically.



The difference $A(r+0.01) - A(r)$ *represents the area of a ring of width* 0.01 *units, where the inner circle forming the ring is a circle with radius* r*. The area of the ring could be approximated by the expression* $0.01 \cdot 2\pi r$*. The average rate of change of the area is* $\dfrac{0.01 \cdot 2\pi r}{0.01} = 2\pi r$*, which is the circumference of the circle with radius* r*.*

Problem Set Sample Solutions

1. Consider the binomial $(2u - 3v)^6$.

   a. Find the term that contains v^4.

   $15(2u)^2(3v)^4 = 4860u^2v^4$

   b. Find the term that contains u^3.

   $20(2u)^3(3v)^3 = 4320u^3v^3$

   c. Find the third term.

   $15(2u)^4(3v)^2 = 2160u^4v^2$

2. Consider the binomial $(u^2 - v^3)^6$.

   a. Find the term that contains v^6.

   $v^6 = (v^3)^2$, $15(u^2)^4(v^3)^2 = 15u^8v^6$

   b. Find the term that contains u^6.

   $u^6 = (u^2)^3$, $20(u^2)^3(v^3)^3 = 20u^6v^9$

   c. Find the fifth term.

   $15(u^2)^2(v^3)^4 = 15u^4v^{12}$

3. Find the sum of all coefficients in the following binomial expansion.

   a. $(2u + v)^{10}$

   $3^{10} = 59049$

   b. $(2u - v)^{10}$

   1

   c. $(2u - 3v)^{11}$

   -1

   d. $(u - 3v)^{11}$

   -2^{11}

   e. $(1 + i)^{10}$

   $((1 + i)^2)^5 = (1 + 2i - 1)^5 = (2i)^5 = 32i$

   f. $(1 - i)^{10}$

   $((1 - i)^2)^5 = (1 - 2i - 1)^5 = (-2i)^5 = -32i$

   g. $(1 + i)^{200}$

   $((1 + i)^2)^{100} = (1 + 2i - 1)^{100} = (2i)^{100} = 2^{100}$

   h. $(1 + i)^{201}$

   $(1 + i)(1 + i)^{200} = (1 + i)((1 + i)^2)^{100} = (1 + i)2^{100}$

4. Expand the binomial $\left(1 + \sqrt{2}i\right)^6$.

$1 + 6(1)^5\left(\sqrt{2}i\right) + 15(1)^4\left(\sqrt{2}i\right)^2 + 20(1)^3\left(\sqrt{2}i\right)^3 + 15(1)^2\left(\sqrt{2}i\right)^4 + 6(1)\left(\sqrt{2}i\right)^5 + \left(\sqrt{2}i\right)^6$

$1 + 6\sqrt{2}i - 30 - 40\sqrt{2}i + 60 + 24\sqrt{2}i - 8 = 23 - 10\sqrt{2}i$

5. Show that $\left(2+\sqrt{2}i\right)^{20}+\left(2-\sqrt{2}i\right)^{20}$ is an integer.

Let's get the first few terms for both to see whether it has any patterns that we can simplify.

$$\left(2+\sqrt{2}i\right)^{2}=2^{20}+A_1(2)^{19}\left(\sqrt{2}i\right)^{1}+A_2(2)^{18}\left(\sqrt{2}i\right)^{2}+A_3(2)^{17}\left(\sqrt{2}i\right)^{3}+A_4(2)^{18}\left(\sqrt{2}i\right)^{4}+A_5(2)^{18}\left(\sqrt{2}i\right)^{5}+\cdots$$
$$-A_{17}(2)^{3}\left(\sqrt{2}i\right)^{17}+A_{18}(2)^{2}\left(\sqrt{2}i\right)^{18}+A_{19}(2)^{1}\left(\sqrt{2}i\right)^{19}+\left(\sqrt{2}i\right)^{20}$$

$$\left(2-\sqrt{2}i\right)^{2}=2^{20}-A_1(2)^{19}\left(\sqrt{2}i\right)^{1}+A_2(2)^{18}\left(\sqrt{2}i\right)^{2}-A_3(2)^{17}\left(\sqrt{2}i\right)^{3}+A_4(2)^{18}\left(\sqrt{2}i\right)^{4}+A_5(2)^{18}\left(\sqrt{2}i\right)^{5}+\cdots$$
$$-A_{17}(2)^{3}\left(\sqrt{2}i\right)^{17}+A_{18}(2)^{2}\left(\sqrt{2}i\right)^{18}-A_{19}(2)^{1}\left(\sqrt{2}i\right)^{19}+\left(\sqrt{2}i\right)^{20}$$

All the even terms are canceled, and the remaining terms would involve multiplying powers of 2 with even powers of $\sqrt{2}$. So, the result would be the sum of some positive and negative integers, which is $116,967,424$.

6. We know $(u+v)^2=u^2+2uv+v^2=u^2+v^2+2uv$. Use this pattern to predict what the expanded form of each expression would be. Then, expand the expression, and compare your results.

   a. $(u+v+w)^2$

      $u^2+v^2+w^2+2uv+2uw+2vw$

   b. $(a+b+c+d)^2$

      $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$

7. Look at the powers of 101 up to the fourth power on a calculator. Explain what you see. Predict the value of 101^5, and then find the answer on a calculator. Are they the same?

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&0&1&&&& \\ &&&1&0&2&0&1&&& \\ &&1&0&3&0&3&0&1&& \\ &1&0&4&0&6&0&4&0&1& \\ 1&0&5&0&10&0&10&0&5&0&1 \end{array}$$

It is similar to Pascal's triangle, but zeros are inserted between the numbers in each row. The answers are not the same; the expanded form of $1|0|5|0|10|0|10|0|5|0|1$ is $10,510,100,501$; however, if we could represent a 10 as a single digit in our base-ten arithmetic, then both answers would be the same.

8. Can Pascal's triangle be applied to $\left(\frac{1}{u}+\frac{1}{v}\right)^n$ given $u, v \neq 0$?

   Yes. The answer is $\left(\frac{1}{u}+\frac{1}{v}\right)^n=\frac{1}{u^n}+\frac{A_1}{u^{(n-1)}v}+\frac{A_2}{u^{(n-2)}v^2}+\cdots+\frac{A_{n-2}}{u^2v^{(n-2)}}+\frac{A_{n-1}}{uv^{(n-1)}}+\frac{1}{v^n}$.

9. The volume and surface area of a sphere are given by $V=\frac{4}{3}\pi r^3$ and $S=4\pi r^2$. Suppose we increase the radius of a sphere by 0.001 units from r to $r+0.001$.

   a. Use the binomial theorem to write an expression for the increase in volume $V(r+0.001)-V(r)$ as the sum of three terms.

$$\begin{aligned} V(r+0.001)-V(r) &= \frac{4}{3}\pi(r+0.001)^3-\frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi[(r)^3+3(r)^2(0.001)+3r(0.001)^2+(0.001)^3]-\frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi r^3+0.004\pi(r)^2+0.000\,004\pi r+\frac{0.000\,000\,004\pi}{3}-\frac{4}{3}\pi r^3 \\ &= 0.004\pi(r)^2+0.000\,004\pi r+\frac{0.000\,000\,004\pi}{3} \end{aligned}$$

b. **Write an expression for the average rate of change of the volume as the radius increases from r to $r + 0.001$.**

Average rate of change: $\dfrac{V(r+0.001)-V(r)}{(r+0.001)-r}$

c. **Simplify the expression in part (b) to compute the average rate of change of the volume of a sphere as the radius increases from r to $r + 0.001$.**

Average rate of change: $\dfrac{0.004\pi(r)^2 + 0.000\,004\pi r + \frac{0.000\,000\,004\pi}{3}}{0.001} = 4\pi r^2 + 0.004\pi r + 0.000\,004\pi$

d. **What does the expression from part (c) resemble?**

Surface area of the sphere with radius r

e. **Why does it make sense that the average rate of change should approximate the surface area? Think about the geometric figure formed by $V(r+0.001) - V(r)$. What does this represent?**

It is a shell of the volume, a layer 0.001 units thick, covering the surface area of the inner sphere of radius r.

f. **How could we approximate the volume of the shell using surface area? And the average rate of change for the volume?**

The volume is approximately $S(r) \cdot 0.001$; the rate of change of volume is approximately $\dfrac{S(r)\times 0.001}{0.001}$, which is the surface area of the sphere, $S(r)$.

g. **Find the difference between the average rate of change of the volume and $S(r)$ when $r = 1$.**

$\text{Average rate of change} = 4\pi(1)^2 + 0.004\pi(1) + 0.000\,004\pi = 12.579$

$S(1) = 4\pi(1)^2 = 12.566$

$12.579 - 12.566 = 0.013$

10. **The area and circumference of a circle of radius r are given by $A(r) = \pi r^2$ and $C(r) = 2\pi r$. Suppose we increase the radius of a sphere by 0.001 units from r to $r + 0.001$.**

a. **Use the binomial theorem to write an expression for the increase in area volume $A(r+0.001) - A(r)$ as a sum of three terms.**

$$\begin{aligned} A(r+0.001) - V(r) &= \pi(r+0.001)^2 - \pi r^2 \\ &= \pi r^2 + 0.002\pi r + 0.000\,001\pi - \pi r^2 \\ &= 0.002\pi r + 0.000\,001\pi \end{aligned}$$

b. **Write an expression for the average rate of change of the area as the radius increases from r to $r + 0.001$.**

Average rate of change: $\dfrac{A(r+0.001)-A(r)}{(r+0.001)-r}$

c. **Simplify the expression in part (b) to compute the average rate of change of the area of a circle as the radius increases from r to $r + 0.001$.**

Average rate of change: $\frac{0.002\pi r + 0.000\,001\pi}{0.001} = 2\pi r + 0.001\pi$

d. **What does the expression from part (c) resemble?**

Surface area of the circle with radius r

e. **Why does it make sense that the average rate of change should approximate the area of a circle? Think about the geometric figure formed by $A(r + 0.001) - A(r)$. What does this represent?**

It is a shell of the volume, a layer 0.01 units thick, covering the surface area of the inner circle of radius r.

f. **How could we approximate the area of the shell using circumference? And the average rate of change for the area?**

The volume is approximately $A(s) \cdot 0.001$; rate of change of volume is approximately $\frac{A(s)\times 0.001}{0.001}$, which is the surface area of the circle, $A(s)$.

g. **Find the difference between the average rate of change of the area and $C(r)$ when $r = 1$.**

$\text{Average rate of change} = 2\pi(1) + 0.001(1) = 6.284$

$C(1) = 2\pi(1) = 6.283$

$6.284 - 6.283 = 0.001$

Lesson 6: Curves in the Complex Plane

Student Outcomes

- Students convert between the real and complex forms of equations for ellipses.
- Students write equations of ellipses and represent them graphically.

Lesson Notes

Initially, students review how to represent numbers in the complex plane using the modulus and argument. They review the characteristics of the graphs of the numbers $= r(\cos(\theta) + i\sin(\theta))$, recognizing that they represent circles centered at the origin with the radius equal to the modulus r. They then explore sets of complex numbers written in the form $z = a\cos(\theta) + bi\sin(\theta)$, identifying the graphs as ellipses. Students convert between the complex and real forms of equations for ellipses, including those whose center is not the origin. They are also introduced to some of the components of ellipses, such as the vertices, foci, and axes. This prepares them to explore ellipses more formally in Lesson 7, where they derive the equation of an ellipse using its foci.

Classwork

Opening Exercise (5 minutes)

This exercise should be completed in pairs or small groups. After a few minutes, students should discuss their responses to Exercises 1–2 with another pair or group before completing Exercise 5. If the students are struggling with how to convert between the rectangular and polar form, the exercises could be completed as part of a teacher-led discussion. Early finishers could display their conjectures and plots for Problem 3, which could be used in a teacher-led discussion of the characteristics of the graph.

Scaffolding:

- For students below grade level, consider a concrete approach using $z = 3 + 2i$, or provide a graphical representation of $z = a + bi$.
- Advanced students could explore the properties of the graph of $z = 3\cos(\theta) + 5i\sin(\theta)$ and compare it to the graph of $z = 5\cos(\theta) + 3i\sin(\theta)$ to form conjectures about the properties of graphs represented by $z = a\cos(\theta) + bi\sin(\theta)$.

Opening Exercise

a. **Consider the complex number $z = a + bi$.**

i. **Write z in polar form. What do the variables represent?**

$z = r(\cos(\theta) + i\sin(\theta))$, where r is the modulus of the complex number and θ is the argument.

ii. **If $r = 3$ and $\theta = 90°$, where would z be plotted in the complex plane?**

The point z is located 3 units above the origin on the imaginary axis.

iii. **Use the conditions in part (ii) to write z in rectangular form. Explain how this representation corresponds to the location of z that you found in part (ii).**

$z = a + bi$, where $a = r\cos(\theta)$ and $b = r\sin(\theta)$

$a = 3\cos(90°) = 0$; $b = 3\sin(90°) = 3$

Then $z = 3i$, which is located three units above the origin on the imaginary axis.

b. **Recall the set of points defined by $z = 3(\cos(\theta) + i\sin(\theta))$ for $0° \le \theta < 360°$, where θ is measured in degrees.**

i. **What does z represent graphically? Why?**

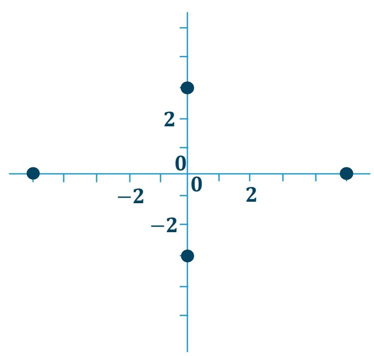
It is the set of points that are 3 units from the origin in the complex plane. This is because the modulus is 3, which indicates that for any given value of θ, z is located a distance of 3 units from the origin.

ii. **What does z represent geometrically?**

A circle with radius 3 units centered at the origin

c. **Consider the set of points defined by $z = 5\cos(\theta) + 3i\sin(\theta)$.**

i. **Plot z for $\theta = 0°, 90°, 180°, 270°, 360°$. Based on your plot, form a conjecture about the graph of the set of complex numbers.**



For $\theta = 0°$, $z = 5\cos(0°) + 3i\sin(0°) = 5 + 0i \leftrightarrow (5, 0)$.

For $\theta = 90°$, $z = 5\cos(90°) + 3i\sin(90°) = 3i \leftrightarrow (0, 3i)$.

For $\theta = 180°$, $z = 5\cos(180°) + 3i\sin(180°) = -5 + 0i \leftrightarrow (-5, 0)$.

For $\theta = 270°$, $z = 5\cos(270°) + 3i\sin(270°) = -3i \leftrightarrow (0, -3i)$.

This set of points seems to form an oval shape centered at the origin.

ii. **Compare this graph to the graph of $z = 3(\cos(\theta) + i\sin(\theta))$. Form a conjecture about what accounts for the differences between the graphs.**

The coefficients of $\cos(\theta)$ and $i\sin(\theta)$ are equal for $z = 3(\cos(\theta) + i\sin(\theta))$, which results in a circle, which has a constant radius, while the coefficients are different for $z = 5\cos(\theta) + 3i\sin(\theta)$, which seems to stretch the circle.

Example 1 (5 minutes)

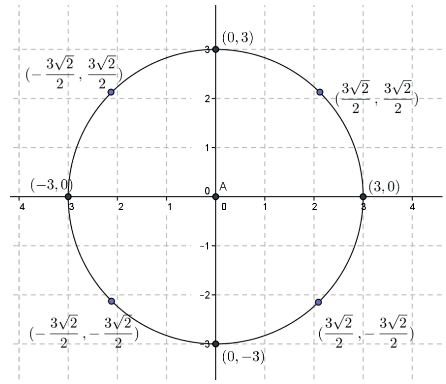
The students are led through an example to demonstrate how to convert the equation of a circle from its complex form to real form (i.e., an equation in x and y). This prepares them to convert the equations of ellipses from complex form to real form.

- We have seen that in the complex plane, the graph of the set of complex numbers defined by $z = 3(\cos(\theta) + i\sin(\theta))$ for $0° \leq \theta < 360°$ is a circle centered at the origin with radius 3 units. How could we represent each point on the circle using an ordered pair?
  - $(3\cos(\theta), 3i\sin(\theta))$
- Let's say we wanted to represent z in the real coordinate plane. We'd need to represent the points on the circle using an ordered pair (x, y). How can we write any complex number z in terms of x and y?
  - $z = x + iy$
- So for $z = 3(\cos(\theta) + i\sin(\theta))$, which expressions represent x and y?
  - $z = 3\cos(\theta) + 3i\sin(\theta)$, *so* $x = 3\cos(\theta)$ *and* $y = 3\sin(\theta)$
- What is the resulting ordered pair?
  - $(3\cos(\theta), 3\sin(\theta))$
- Let's graph some points and verify that this gives us a circle. Complete the table for the given values of θ.

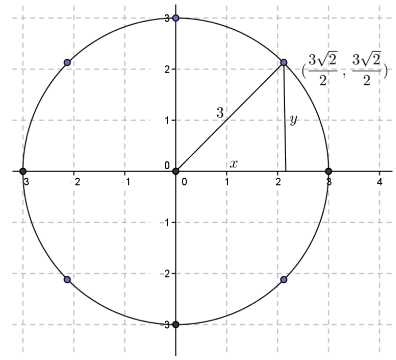
Example 1

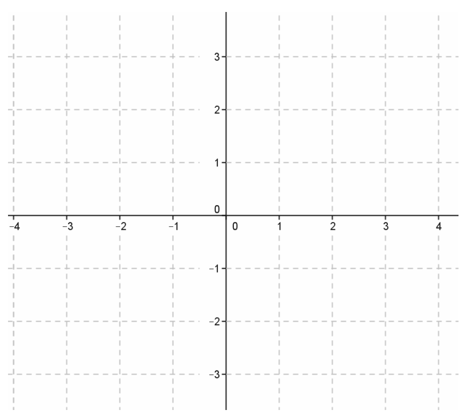
Consider again the set of complex numbers represented by $z = 3(\cos(\theta) + i\sin(\theta))$ for $0° \leq \theta < 360°$.

θ	$3\cos(\theta)$	$3\sin(\theta)$	$(3\cos(\theta), 3i\sin(\theta))$
0	3	0	$(3, 0)$
$\frac{\pi}{4}$	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	0	3	$(0, 3)$
$\frac{3\pi}{4}$	$-\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$
π	-3	0	$(-3, 0)$
$\frac{5\pi}{4}$	$-\frac{3\sqrt{2}}{2}$	$-\frac{3\sqrt{2}}{2}$	$\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	0	-3	$(0, -3)$
$\frac{7\pi}{4}$	$\frac{3\sqrt{2}}{2}$	$-\frac{3\sqrt{2}}{2}$	$\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$
2π	3	0	$(3, 0)$



- Plot the points in the table, and determine the type of curve created.
  - *The curve is a circle with center* $(0, 0)$ *and radius* 3.
- If $x = 3\cos(\theta)$ and $y = 3\sin(\theta)$, write an equation that relates x^2 and y^2.
  - $x^2 + y^2 = (3\cos(\theta))^2 + (3\sin(\theta))^2$
- Now, simplify the right side of that equation.
  - $x^2 + y^2 = 9\cos^2(\theta) + 9\sin^2(\theta)$
  - $x^2 + y^2 = 9(\cos^2(\theta) + \sin^2(\theta))$
- Do you know a trigonometric identity that relates $\sin^2(\theta)$ and $\cos^2(\theta)$?
  - $\cos^2(\theta) + \sin^2(\theta) = 1$
- Substitute the identity into the previous equation.
  - $x^2 + y^2 = 9$
- How does the graph of this equation compare with the graph of our equation in complex form?
  - *Both graphs are circles centered at the origin with radius* 3 *units.*





a. **Use an ordered pair to write a representation for the points defined by z as they would be represented in the coordinate plane.**

$(3\cos(\theta), 3\sin(\theta))$

b. **Write an equation that is true for all the points represented by the ordered pair you wrote in part (a).**

Since $x = 3\cos(\theta)$ ***and*** $y = 3\sin(\theta)$***:***

$$x^2 + y^2 = (3\cos(\theta))^2 + (3\sin(\theta))^2$$
$$x^2 + y^2 = 9(\cos(\theta))^2 + 9(\sin(\theta))^2$$
$$x^2 + y^2 = 9((\cos(\theta))^2 + (\sin(\theta))^2)$$

We know that $(\sin(\theta))^2 + (\cos(\theta))^2 = 1$***, so*** $x^2 + y^2 = 9$.

c. **What does the graph of this equation look like in the coordinate plane?**

The graph is a circle centered at the origin with radius 3 units.

Exercises 1–2 (6 minutes)

The students should complete the exercises independently. After a few minutes, they could verify their responses with a partner. The solutions should be reviewed in a whole-class setting after students have had a sufficient amount of time to complete the exercises.

Scaffolding:

If students are struggling with converting the equations into real form, suggest that they square x and y and then isolate $\cos^2(\theta)$ or $\sin^2(\theta)$ in the equations. Alternatively, Exercise 1 could be completed as guided practice, and the students could then complete Exercise 2 independently.

Exercises 1–2

1. **Recall the set of points defined by $z = 5\cos(\theta) + 3i\sin(\theta)$.**

   a. **Use an ordered pair to write a representation for the points defined by z as they would be represented in the coordinate plane.**

   $(5\cos(\theta), 3\sin(\theta))$

   b. **Write an equation in the coordinate plane that is true for all the points represented by the ordered pair you wrote in part (a).**

   We have $x = 5\cos(\theta)$ and $y = 3\sin(\theta)$, so $x^2 = 25(\cos^2(\theta))$ and $y^2 = 9(\sin^2(\theta))$. We know $(\cos^2(\theta) + \sin^2(\theta)) = 1$.

   Since $x^2 = 25(\cos^2(\theta))$, then $\cos^2(\theta) = \frac{x^2}{25}$. Since $y^2 = 9(\sin^2(\theta))$, then $\sin^2(\theta) = \frac{y^2}{9}$. By substitution, $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

2. **Find an algebraic equation for all the points in the coordinate plane traced by the complex numbers $z = \sqrt{2}\cos(\theta) + i\sin(\theta)$.**

   All the complex numbers represented by z can be written using the ordered pair $\left(\sqrt{2}\cos(\theta), \sin(\theta)\right)$ in the coordinate plane.

   We have $x = \sqrt{2}\cos(\theta)$ and $y = \sin(\theta)$, so $x^2 = 2\cos^2(\theta)$ and $y^2 = \sin^2(\theta)$. We know $\cos^2(\theta) + \sin^2(\theta) = 1$.

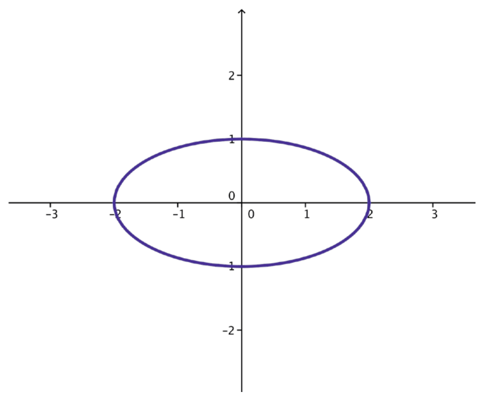
   Since $x^2 = 2\cos^2(\theta)$, then $\cos^2(\theta) = \frac{x^2}{2}$. Then, by substitution, $\frac{x^2}{2} + y^2 = 1$.

Discussion (5 minutes): Describing an Ellipse

- At the outset of the lesson, we determined that the graph of the complex numbers defined by $z = 5\cos\theta + 3i\sin\theta$ was an oval shape that was centered about the origin and intersected the axes at the points $(5, 0)$, $(-5, 0)$, $(0, 3i)$, and $(0, -3i)$.

MP.3

- Make a conjecture about the graph of the complex numbers defined by $z = \sqrt{2}\cos(\theta) + i\sin(\theta)$. Sketch a rough graph of the points to test your conjecture. Share and discuss your conjecture with a neighbor. (Pull the class back together to debrief.)
  - *The graph would be a closed curve centered at the origin that intersects the axes at the points* $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$, $(0, 1)$, *and* $(0, -1)$.

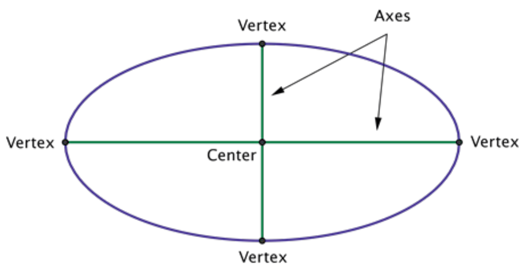


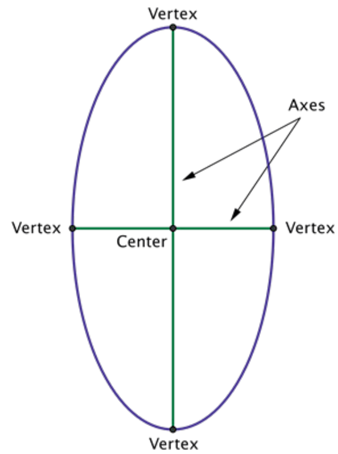
- What patterns do you notice between the graphs we have sketched and the structure of their equations in real form?
  - *The equations are written in the form of* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, *where* $|a|$ *is the distance from the origin to the* x*-intercepts and* $|b|$ *is the distance from the* y*-intercepts.*

MP.7

- And what patterns do you notice between the graphs we have sketched and the structure of their equations in complex form?
  - a *is the coefficient of* $\cos(\theta)$, *and* b *is the coefficient of* $i\sin(\theta)$.
- Let's formalize these observations. The shape that arises from the curve given by an equation in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is called an *ellipse centered at the origin*. An ellipse can be stretched horizontally or vertically, as shown in the figure on the following page.

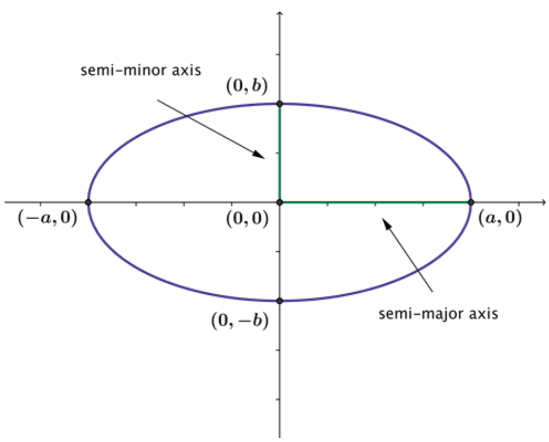
> *Scaffolding:*
>
> Have students complete a Frayer diagram for an ellipse. An example can be found in Module 1 Lesson 5.

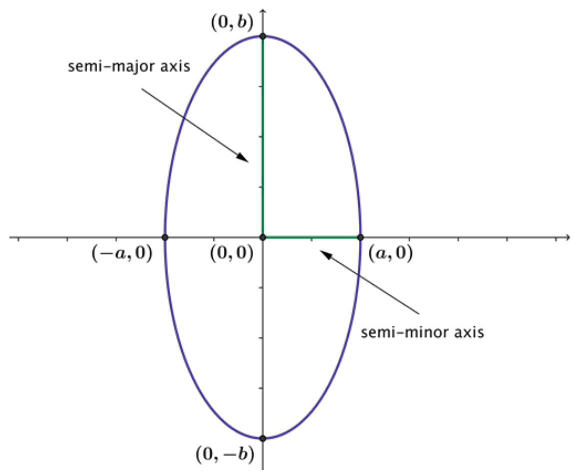




The *vertices* of the ellipse are the four points on the ellipse that are the closest to and farthest from the center. The *axes* of the ellipse are the segments connecting opposite vertices. The *major axis* is the longer of the two axes, and the *minor axis* is the shorter of the two axes. In the ellipse shown to the left above, the major axis is horizontal; in the ellipse shown to the right, the major axis is vertical.

The *semi-major axis* is defined as a segment between the center of the ellipse and a vertex along the major axis, and the *semi-minor axis* is a segment between the center of the ellipse and a vertex along the minor axis.





- If the ellipse is centered at the origin, then the vertices of the ellipse are the intercepts $(-a, 0)$, $(a, 0)$, $(0, -b)$, and $(0, b)$. In this case, what is the length of the semi-major axis?
  - *Either $|a|$ or $|b|$, whichever is larger*
- Good. For an ellipse centered at the origin, what is the length of the semi-minor axis?
  - *Either $|a|$ or $|b|$, whichever is smaller*
- And what happens when $a = b$? Explain how you know.
  - *We get a circle because the distances from the center to a and b are the same.*

Example 2 (5 minutes)

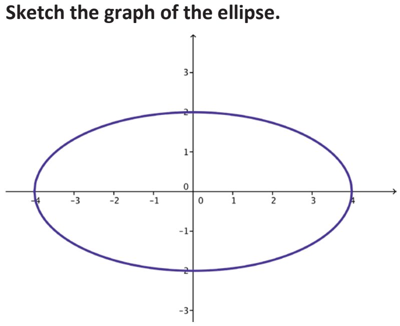
This example prepares students to sketch the graphs of ellipses from their equations written in real form. It also demonstrates how to convert equations of ellipses from real to complex form.

- How can we tell that an algebraic equation represents an ellipse without being told?
  - *It can be written in the form* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- And how can we use what we know about a and b to plot the ellipse?
  - $|a|$ *represents the distance from the center of the ellipse to the points to the right and left of its center, so the graph of our equation intersects the* x*-axis at* $(4, 0)$ *and* $(-4, 0)$. *The number* $|b|$ *represents the distance from the center of the ellipse to the points above and below its center, so the graph intersects the* y*-axis at* $(0, 2)$ *and* $(0, -2)$.
- Now, let's write the equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$ in complex form. What is the structure of a general equation for an ellipse in complex form?
  - $z = a\cos(\theta) + bi\sin(\theta)$
- What do we need to find, then, to write the equation in complex form?
  - *Values of* a *and* b
- How could we find values of a and b?
  - *Since* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *we have* $16 = a^2$ *and* $4 = b^2$, *which means* $a = 4$ *and* $b = 2$.
- What is the complex form of the equation of the ellipse given by $\frac{x^2}{16} + \frac{y^2}{4} = 1$?
  - $z = 4\cos(\theta) + 2i\sin(\theta)$

Example 2

The equation of an ellipse is given by $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

a. Sketch the graph of the ellipse.



b. **Rewrite the equation in complex form.**

$\frac{x^2}{16} + \frac{y^2}{4} = 1$

Since $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$*, we have* $a = 4$ *and* $b = 2$.

The complex form of the ellipse is $z = a\cos(\theta) + bi\sin(\theta) = 4\cos(\theta) + 2i\sin(\theta)$.

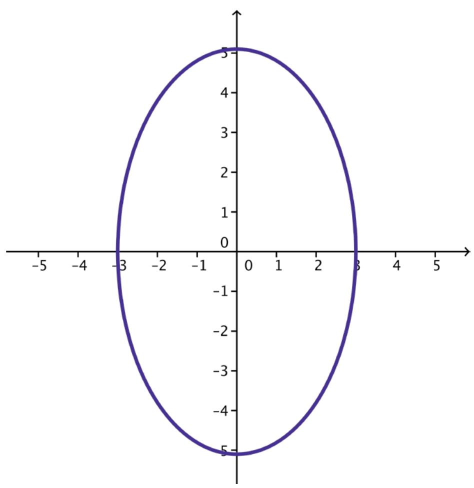
Exercise 3 (4 minutes)

Students should complete the exercise in pairs. They should solve the problem independently and, after a few minutes, verify their solutions with a partner. At an appropriate time, pairs of selected students should share their sketches or the complex form for the equation. Make sure the students recognize the change in the orientation of the ellipse (i.e., that it is elongated vertically because $b > a$).

Exercise 3

3. **The equation of an ellipse is given by** $\frac{x^2}{9} + \frac{y^2}{26} = 1$.

   a. **Sketch the graph of the ellipse.**



   b. **Rewrite the equation of the ellipse in complex form.**

$\frac{x^2}{9} + \frac{y^2}{26} = 1$

$|a| = 3$

$|b| = \sqrt{26}$

The complex form of the ellipse is $z = a\cos(\theta) + bi\sin(\theta) = 3\cos(\theta) + \sqrt{26}i\sin(\theta)$.

Example 3 (5 minutes)

This example introduces students to ellipses that are not centered at the origin and prepares them to convert translated ellipses from complex to real form and to sketch their graphs.

- How does this equation look different from others we have seen in this lesson?
  - *There are constants included that were not in the other equations.*
- How can we represent the complex numbers z in rectangular form?
  - $z = x + iy$
- And what are the values of x and y for this equation?
  - $x = 2 + 7\cos(\theta)$ *and* $y = 1 + \sin(\theta)$
- What is our procedure for converting equations of ellipses from complex to real form?
  - *Isolate* $\cos(\theta)$ *and* $\sin(\theta)$, *and then substitute the equivalent expressions into the equation* $\cos^2(\theta) + \sin^2(\theta) = 1$.
- What is the resulting equation?
  - $\frac{(x-2)^2}{49} + (y-1)^2 = 1$
- How do we know this equation represents an ellipse?
  - *It is written in the form* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- What are the values of a and b?
  - $a = 7$ *and* $b = 1$
- How do the constants subtracted from x and y affect the graph of the ellipse?
  - *They represent a translation of the center from the origin* 2 *units to the right and* 1 *unit up.*
- Describe the graph of the ellipse.
  - *The ellipse is centered at* $(2, 1)$ *and is elongated horizontally, so the semi-major axis has length* 7 *units, and the semi-minor axis has length* 1 *unit.*

Example 3

A set of points in the complex plane can be represented in the complex plane as $z = 2 + i + 7\cos(\theta) + i\sin(\theta)$ as θ varies.

a. Find an algebraic equation for the points described.

$z = 2 + i + 7\cos(\theta) + i\sin(\theta) = (2 + 7\cos(\theta)) + i(1 + \sin(\theta))$

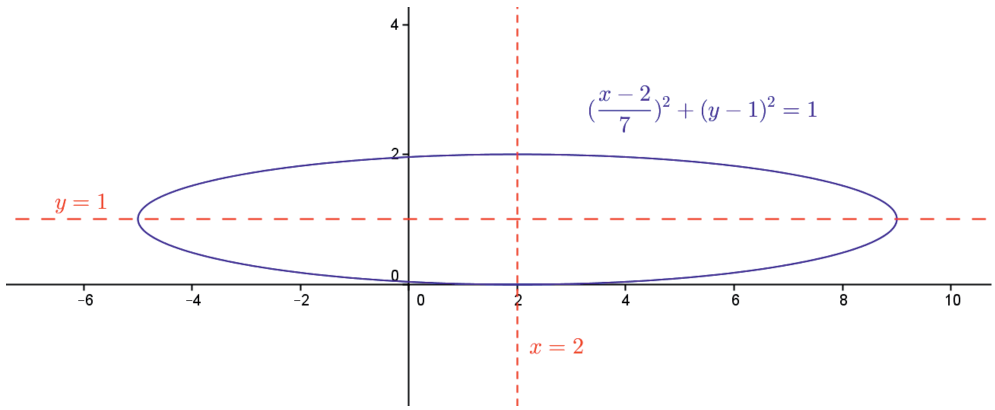
***Since** $z = x + iy$, **then** $x = 2 + 7\cos(\theta)$ **and** $y = 1 + \sin(\theta)$.*

***So** $\cos(\theta) = \frac{x-2}{7}$ **and** $\sin(\theta) = (y-1)$.*

Since** $\cos^2(\theta) + \sin^2(\theta) = 1$, **we have** $\left(\frac{x-2}{7}\right)^2 + (y-1)^2 = 1$, **which is equivalent to

$\frac{(x-2)^2}{49} + (y-1)^2 = 1.$

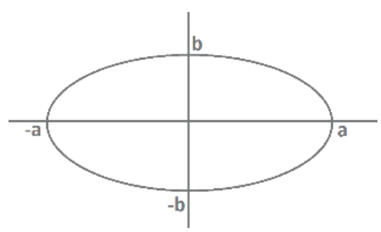
b. Sketch the graph of the ellipse.



Closing (5 minutes)

Have the students summarize the information on ellipses. As a class, a list of the key features of an ellipse can be compiled and displayed. A list of key features should address:

- An ellipse is a curve that represents the set of complex numbers that satisfy the equation $z = a\cos(\theta) + bi\sin(\theta)$ for $0° \leq \theta < 360°$.
- If $a = b$, the curve is a circle with radius r, and the equation can be simplified to $z = r(\cos(\theta) + i\sin(\theta))$.
- In the coordinate plane, ellipses centered at the origin can be represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $|a|$ is the half of the length of the horizontal axis and $|b|$ is the half of the length of the vertical axis.
- The sketch of a general ellipse is



- An ellipse is elongated horizontally if $a > b$ and elongated vertically when $b > a$.
- The points on an ellipse can be written in polar form as $(a\cos(\theta), b\sin(\theta))$.
- An ellipse with center (h, k) can be represented by the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

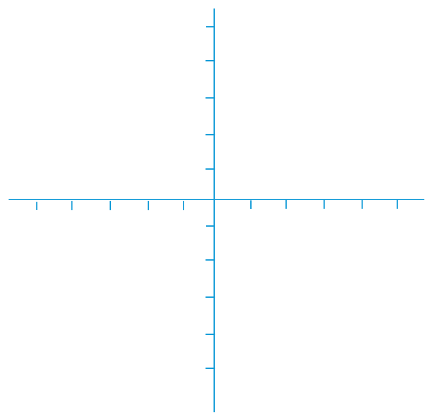
Exit Ticket (5 minutes)

Name ______________________________ Date ______________

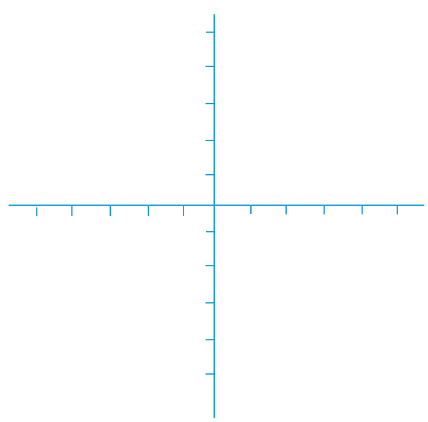
Lesson 6: Curves in the Complex Plane

Exit Ticket

1. Write the real form of the complex equation $z = \cos(\theta) + 3i\sin(\theta)$. Sketch the graph of the equation.



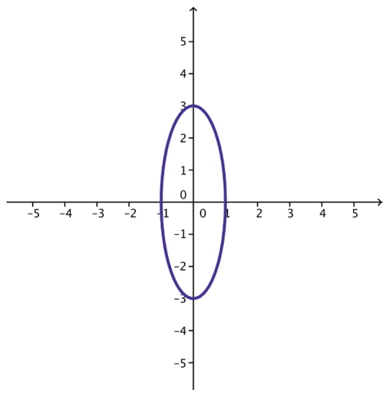
2. Write the complex form of the equation $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Sketch the graph of the equation.



Exit Ticket Sample Solutions

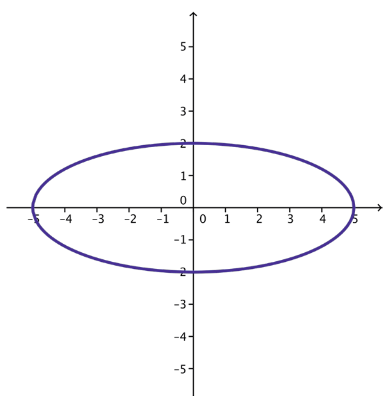
1. **Write the real form of the complex equation $z = \cos(\theta) + 3i\sin(\theta)$. Sketch the graph of the equation.**

   $x^2 + \frac{y^2}{9} = 1$



2. **Write the complex form of the equation $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Sketch the graph of the equation.**

   $z = 5\cos(\theta) + 2i\sin(\theta)$



Problem Set Sample Solutions

Problem 6 is an extension that requires students to convert an algebraic equation for an ellipse to standard form. The problem could be presented using the standard form of the equation (the answer for part (a)) to provide students with additional practice converting the equations of ellipses between complex and real forms.

1. **Write the real form of each complex equation.**
   a. $z = 4\cos(\theta) + 9i\sin(\theta)$

      $\frac{x^2}{16} + \frac{y^2}{81} = 1$

b. $z = 6\cos(\theta) + i\sin(\theta)$

$$\frac{x^2}{36} + y^2 = 1$$

c. $z = \sqrt{5}\cos(\theta) + \sqrt{10}i\sin(\theta)$

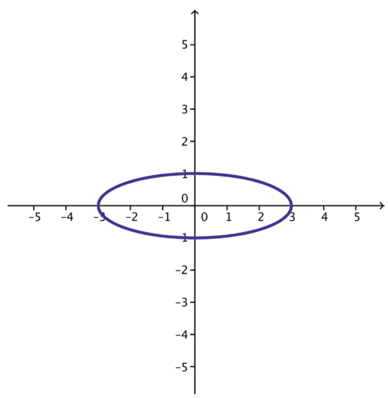
$$\frac{x^2}{5} + \frac{y^2}{10} = 1$$

d. $z = 5 - 2i + 4\cos(\theta) + 7i\sin(\theta)$

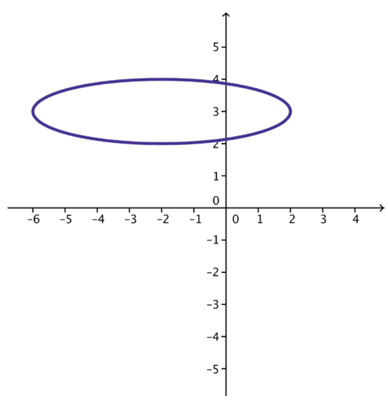
$$\frac{(x-5)^2}{16} + \frac{(y+2)^2}{49} = 1$$

2. **Sketch the graphs of each equation.**

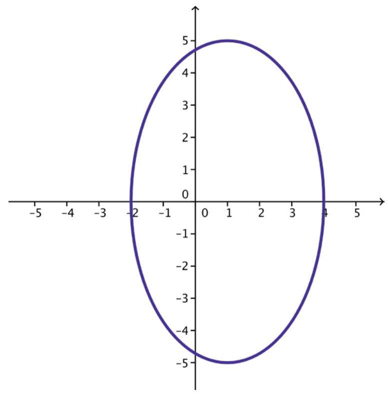
a. $z = 3\cos(\theta) + i\sin(\theta)$



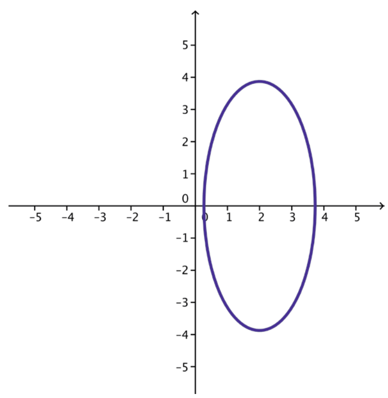
b. $z = -2 + 3i + 4\cos(\theta) + i\sin(\theta)$



c. $\frac{(x-1)^2}{9} + \frac{y^2}{25} = 1$



d. $\frac{(x-2)^2}{3} + \frac{y^2}{15} = 1$



3. **Write the complex form of each equation.**

a. $\frac{x^2}{16} + \frac{y^2}{36} = 1$

$z = 4\cos(\theta) + 6i\sin(\theta)$

b. $\frac{x^2}{400} + \frac{y^2}{169} = 1$

$z = 20\cos(\theta) + 13i\sin(\theta)$

c. $\frac{x^2}{19} + \frac{y^2}{2} = 1$

$z = \sqrt{19}\cos(\theta) + \sqrt{2}i\sin(\theta)$

d. $\frac{(x-3)^2}{100} + \frac{(y+5)^2}{16} = 1$

$z = 3 - 5i + 10\cos(\theta) + 4i\sin(\theta)$

4. Carrie converted the equation $z = 7\cos(\theta) + 4i\sin(\theta)$ to the real form $\frac{x^2}{7} + \frac{y^2}{4} = 1$. Her partner Ginger said that the ellipse must pass through the point $(7\cos(0), 4\sin(0)) = (7, 0)$ and this point does not satisfy Carrie's equation, so the equation must be wrong. Who made the mistake, and what was the error? Explain how you know.

Ginger is correct. Carrie set $a = 7$ and $b = 4$, which is correct, but then she made an error in converting to the real form of the equation by dividing by a and b instead of a^2 and b^2.

5. Cody says that the center of the ellipse with complex equation $z = 4 - 5i + 2\cos(\theta) + 3i\sin(\theta)$ is $(4, -5)$, while his partner, Jarrett, says that the center of this ellipse is $(-4, 5)$. Which student is correct? Explain how you know.

Cody is correct. This ellipse is the translation of the ellipse with equation $z = 2\cos(\theta) + 3i\sin(\theta)$ by the vector $\langle 4, -5\rangle$, which moves the center of the ellipse from the origin to the point $(4, -5)$.

Extension:

6. Any equation of the form $ax^2 + bx + cy^2 + dy + e = 0$ with $a > 0$ and $c > 0$ might represent an ellipse. The equation $4x^2 + 8x + 3y^2 + 12y + 4 = 0$ is such an equation of an ellipse.

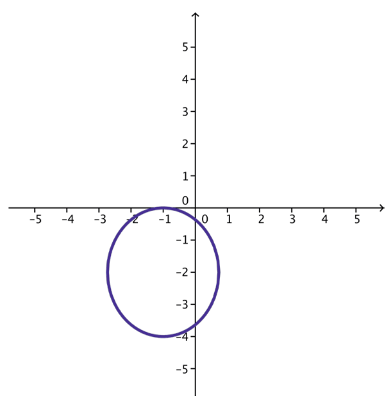
   a. Rewrite the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ in standard form to locate the center of the ellipse (h, k).

$$4(x^2 + 2x) + 3(y^2 + 4y) + 4 = 0$$
$$4(x^2 + 2x + 1) + 3(y^2 + 4y + 4) = -4 + 4(1) + 3(4)$$
$$4(x + 1)^2 + 3(y + 2)^2 = 12$$
$$\frac{4(x + 1)^2}{12} + \frac{3(y + 2)^2}{12} = 1$$
$$\frac{(x + 1)^2}{3} + \frac{(y + 2)^2}{4} = 1$$

   The center of the ellipse is the point $(-1, -2)$.

   b. Describe the graph of the ellipse, and then sketch the graph.

   The graph of the ellipse is centered at $(-1, -2)$. It is elongated vertically with a semi-major axis of length 2 units and a semi-minor axis of length $\sqrt{3}$ units.



c. **Write the complex form of the equation for this ellipse.**

$$\frac{(x+1)^2}{3}+\frac{(y+2)^2}{4}=1$$

$\cos^2(\theta)+\sin^2(\theta)=1$, *so* $\cos^2(\theta)=\frac{(x+1)^2}{3}$ *and* $\sin^2(\theta)=\frac{(y+2)^2}{4}$

$3\cos^2(\theta)=(x+1)^2$, *so* $x=\sqrt{3}\cos(\theta)-1$

$4\sin^2(\theta)=(y+2)^2$, *so* $y=2\sin(\theta)-2$

$$\begin{aligned} z &= x+iy \\ &= \sqrt{3}\cos(\theta)-1+i(2\sin(\theta)-2) \\ &= -1-2i+\sqrt{3}\cos(\theta)+2i\sin(\theta) \end{aligned}$$

Lesson 7: Curves from Geometry

Student Outcomes

- Students derive the equations of ellipses given the foci, using the fact that the sum of distances from the foci is constant (**G-GPE.A.3**).

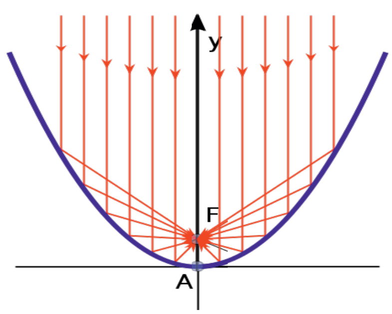
Lesson Notes

In the previous lesson, students encountered sets of points that satisfy an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. In this lesson, students are introduced to ellipses as sets of points such that each point P satisfies the condition $PF + PG = k$, where points F and G are the foci of the ellipse and k is a constant. The goal of this lesson is to connect these two representations. That is, if a point $P(x, y)$ satisfies the condition $PF + PG = k$, then it also satisfies an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for suitable values of a and b.

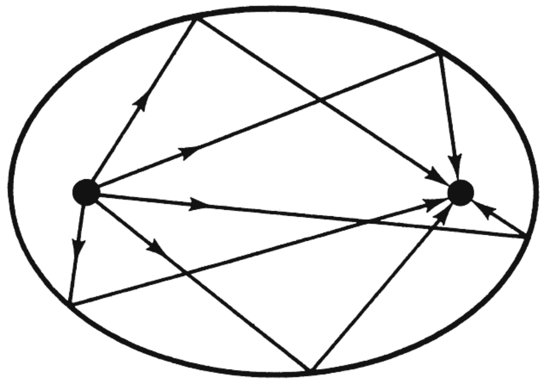
Classwork

Opening (8 minutes)

- In the previous lesson, we learned that an ellipse is an oval-shaped curve that can arise in the complex plane. In this lesson, we continue our analysis of elliptical curves. But first, let's establish a connection to another curve that we studied in Algebra II.
- In Algebra II, we learned that parabolic mirrors are used in the construction of telescopes. Do you recall the key property of parabolic mirrors? Every light ray that is parallel to the axis of the parabola reflects off the mirror and is sent to the same point, the focus of the parabola.

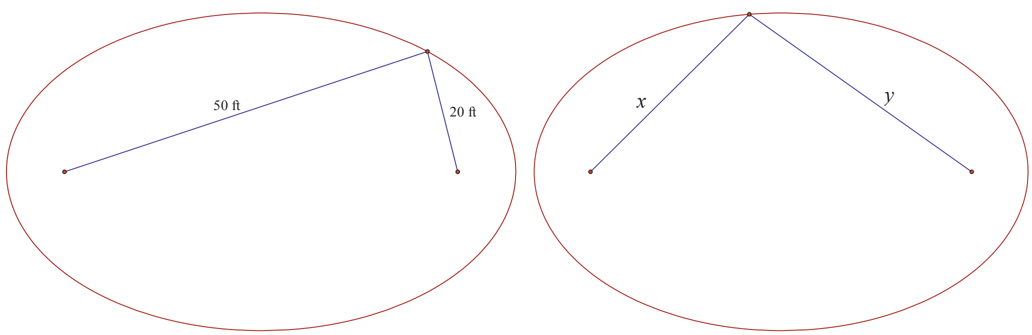


- Like the parabola, the ellipse has some interesting reflective properties as well. Imagine that you and a friend are standing at the focal points of an elliptical room, as shown in the diagram below. Though your friend may be 100 feet away, you would be able to hear what she is saying, even if she were facing away from you and speaking at the level of a whisper! How can this be? This phenomenon is based on the reflective property of ellipses: Every ray emanating from one focus of the ellipse is reflected off the curve in such a way that it travels to the other focal point of the ellipse.



- A famous example of a room with this curious property is the National Statuary Hall in the United States Capitol. Can you see why people call this chamber "the Whispering Gallery?" The brief video clip below serves to illustrate this phenomenon.

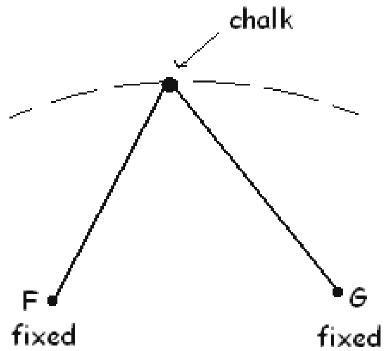
  https://www.youtube.com/watch?v=FX6rUU_74kk
- Now, let's turn our attention to the mathematical properties of elliptical rooms. If every sound wave emanating from one focus bounces off the walls of the room and reaches the other focus at exactly the same instant, what can we say about the lengths of the segments shown in the following two diagrams?



- In the second figure, the sound is directed toward a point that is closer to the focus on the left but farther from the focus on the right. So, it appears that x is less than 50 and y is more than 20. Can we say anything more specific than this? Think about this for a moment, and then share your thoughts with a neighbor.

MP.7

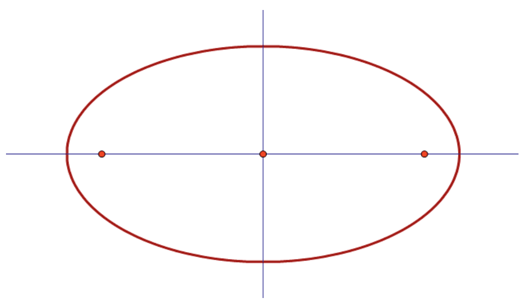
- Draw as many segments as you can from one focus, to the ellipse, and back to the other focus. Share what you notice with a neighbor. (Debrief as a class.)
  - *As one segment gets longer, the other gets shorter.*
  - *The total length of the two segments seems to be roughly the same.*
- If the sound waves travel from one focus of the ellipse to the other in the same period of time, it follows that they must be traveling equal distances! So, while we can't say for sure what the individual distances x and y are, we do know that $x + y = 70$. This leads us to the distance property of ellipses: For any point on an ellipse, the sum of the distances to the foci is constant.
- Here is a concrete demonstration of the distance property for an ellipse. Take a length of string, and attach the two ends to the board. Then, place a piece of chalk on the string, and pull it taut. Move the chalk around the board, keeping the string taut. Because the string has a fixed length, the sum of the distances to the foci remains the same. Thus, this drawing technique generates an ellipse.



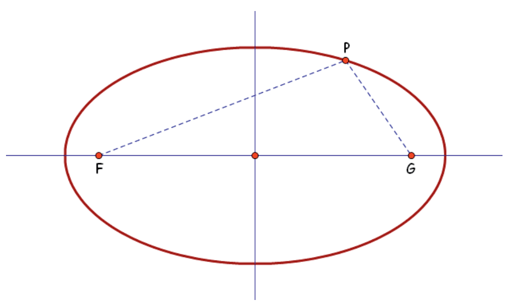
- In the previous lesson, we viewed an ellipse centered at the origin as a set of points that satisfy an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Do you think that a room with the reflection property discussed above can be described by this kind of equation? Let's find out together.

Discussion (16 minutes)

- Since we are going to describe an ellipse using an equation, let's bring coordinate axes into the picture.
- Let's place one axis along the line that contains the two focal points, and let's place the other axis at the midpoint of the two foci.



- Suppose that the focal points are located 8 feet apart, and a sound wave traveling from one focus to the other moves a total distance of 10 feet. Exactly which points in the plane are on this elliptical curve? Perhaps it helps at this stage to add in a few labels.

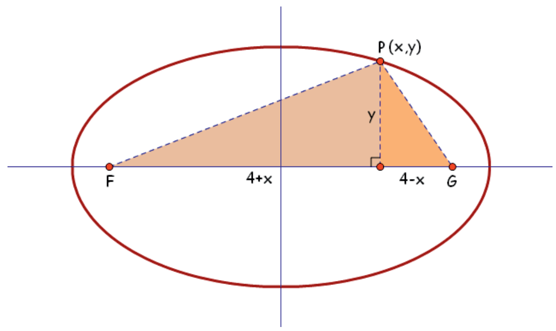


MP.2

- Can you represent the distance condition that defines this ellipse into an equation involving these symbols?
  - *Since we were told that sound waves travel 10 feet from one focal point to the other, we need to have $PF + PG = 10$.*

MP.2

- What can we say about the coordinates of P? In other words, what does it take for a point (x, y) to be on this ellipse? First of all, we were told that the foci were 8 feet apart, so the coordinates of the foci are $(-4,0)$ and $(4,0)$. Can you represent the condition $PF + PG = 10$ as an equation about x and y? The picture below may help you to do this.



  - *The distance from points F and G to the y-axis is 4. If point P is a distance of x from the y-axis, that means the horizontal distance from P to F is $4 + x$ and from P to G is $4 - x$.*
  - *The distance PF is given by $\sqrt{(4+x)^2 + y^2}$, and the distance PG is given by $\sqrt{(4-x)^2 + y^2}$. Since the sum of these two distances must be 10, the coordinates of P must satisfy the equation* $\sqrt{(4+x)^2 + y^2} + \sqrt{(4-x)^2 + y^2} = 10$.

Scaffolding:

- In order to help students construct radical expressions such as $\sqrt{(4+x)^2 + y^2}$ and $\sqrt{(4-x)^2 + y^2}$, prompt them to recall the Pythagorean theorem and its meaning.
- In the context of the Pythagorean theorem, what do a, b, and c represent in $a^2 + b^2 = c^2$?
- How could this equation be transformed so that we have an equation for c in terms of a and b?
- What expressions could be substituted for a, b, and c in this example?

- Now that it's done, we can set about verifying that a curve with this special distance property can be written in the very simple form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It looks like the first order of business is to get rid of the square roots in the equation. How should we go about that?
  - *We can eliminate the square roots by squaring both sides.*
- Experience shows that we have a much easier time of it if we put one of the radical expressions on the other side of the equation first. (Try squaring both sides without doing this, and you quickly see the wisdom of this approach!) Let's subtract the second radical expression from both sides:

$$\sqrt{(4+x)^2 + y^2} = 10 - \sqrt{(4-x)^2 + y^2}$$

- Now, we are all set to square both sides:

$$(4+x)^2 + y^2 = 100 - 20\sqrt{(4-x)^2 + y^2} + (4-x)^2 + y^2$$

- As usual, we have a variety of choices about how to proceed. What ideas do you have?
  - *We can subtract y^2 from both sides, giving*

    $(4+x)^2 = 100 - 20\sqrt{(4-x)^2 + y^2} + (4-x)^2$.
  - *We can expand the binomials, giving*

    $16 + 8x + x^2 = 100 - 20\sqrt{(4-x)^2 + y^2} + 16 - 8x + x^2$.
- Now what?
  - *We can subtract 16 and x^2 from both sides of this equation, giving*

    $8x = 100 - 20\sqrt{(4-x)^2 + y^2} - 8x$.

- We started out with two radical expressions, and we have managed to get down to one. That is progress, but we want to square both sides again to eliminate all radicals from the equation. With that goal in mind, it is helpful to put the radical expression on one side of the equation and everything else on the other side, like this:

$$20\sqrt{(4-x)^2+y^2} = 100 - 16x$$

- Now, we square both sides:

$$400[(4-x)^2+y^2] = 10000 - 3200x + 256x^2$$

- Expanding the binomial gives

$$400[16-8x+x^2+y^2] = 10000 - 3200x + 256x^2$$
$$6400 - 3200x + 400x^2 + 400y^2 = 10000 - 3200x + 256x^2$$
$$144x^2 + 400y^2 = 3600$$

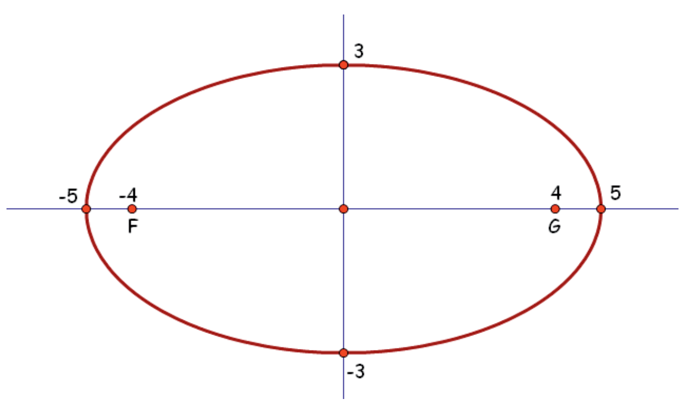
- We began with an equation that contained two radical expressions and generally looked messy. After many algebraic manipulations, things are looking a whole lot simpler. There is just one further step to whip this equation into proper shape, that is, to check that this curve indeed has the form of an ellipse, namely, $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$. Can you see what to do?
  - *We just divide both sides of the equation by* $3{,}600$*:*

$$\frac{x^2}{25}+\frac{y^2}{9} = 1$$

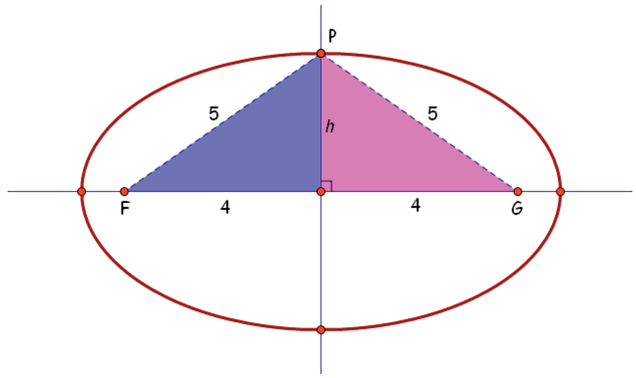
- Here is a recap of our work up to this point: An ellipse is a set of points where the sum of the distances to two fixed points (called *foci*) is constant. We studied a particular example of such a curve, showing that it can be written in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

$$\frac{x^2}{25}+\frac{y^2}{9} = 1$$

- Let's see what else we can learn about the graph of this ellipse by looking at this equation. Where does the curve intersect the axes?
  - *You can tell by inspection that the curve contains the points* $(5,0)$ *and* $(-5,0)$*, as well as the points* $(0,3)$ *and* $(0,-3)$*.*



- Now, let's verify that these features are consistent with the facts we started with, that is, that the foci were 8 feet apart and that the distance each sound wave travels between the foci is 10 feet. Do the x-intercepts make sense in light of these facts? Think about this, and then share your response with a neighbor.
  - *Yes, the intercepts are $(-5,0)$ and $(5,0)$. If a person is standing at F and another person is standing at G, and the person at point F speaks while facing left, the sound travels 1 foot to the wall and then bounces 9 feet back to G. The sound travels a total of 10 feet.*
- Next, let's check to see if the y-intercepts are consistent with the initial conditions of this problem.

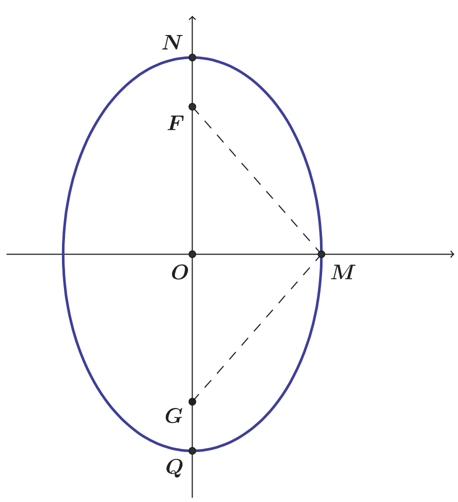


  - *It looks as though the y-axis is a line of symmetry for this ellipse, so we should expect that the y-intercept is a point P on the elliptical wall where the distance from P to F is the same as the distance from P to G. Since the total distance is 10 feet, we must have $PF = PG = 5$. In that case, the height of the triangles in the picture must satisfy the Pythagorean theorem so that $h^2 + 4^2 = 5^2$. It is clear that the equation is true when $h = 3$, so one of the y-intercepts is indeed $(0,3)$.*

Example (5 minutes)

In this example, students derive the equation of an ellipse whose foci lie on the y-axis.

- Tammy takes an 8-inch length of string and tapes the ends to a chalkboard in such a way that the ends of the string are 6 inches apart. Then, she pulls the string taut and traces the curve below using a piece of chalk.



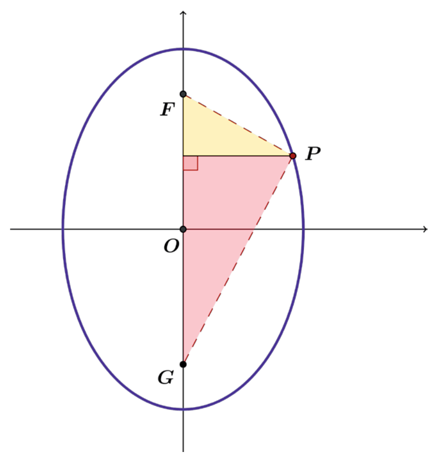
- Where does this curve intersect the y-axis? Take a minute to think about this.
  - *When the chalk is at point N, the string goes from F to N, then back down to G. We know that $FG = 6$, and the string is 8 inches long, so we have $FN + NG = 8$, which means $FN + (FN + 6) = 8$, which means FN must be 1.*
  - *Thus, point F is $(0,3)$, and point N is $(0,4)$. Similarly, point G is $(0,-3)$, and point Q is at $(0,-4)$.*
- So, where does the curve intersect the x-axis? Use the diagram to help you.
  - *When the chalk is at point M, the string is divided into two equal parts. Thus, $FM = 4$. The length OM must satisfy $(OM)^2 + 3^2 = 4^2$, so $OM = \sqrt{7}$.*

Exercise (6 minutes)

In this exercise, students practice rewriting an equation involving two radical expressions. This is the core algebraic work involved in meeting standard **G-GPE.A.3** as it relates to ellipses.

Exercise

Points F and G are located at $(0,3)$ and $(0,-3)$. Let $P(x,y)$ be a point such that $PF + PG = 8$. Use this information to show that the equation of the ellipse is $\frac{x^2}{7} + \frac{y^2}{16} = 1$.



The distance from the x-axis to point F and to point G is 3. The distance from the x-axis to point P is x; that means the vertical distance from F to P is $3 - y$, and the vertical distance from G to P is $3 + y$.

$$PF + PG = \sqrt{x^2 + (3-y)^2} + \sqrt{x^2 + (y+3)^2} = 8$$
$$\sqrt{x^2 + (3-y)^2} = 8 - \sqrt{x^2 + (y+3)^2}$$
$$x^2 + (3-y)^2 = 64 - 16\sqrt{x^2 + (y+3)^2} + x^2 + (y+3)^2$$
$$y^2 - 6y + 9 = 64 - 16\sqrt{x^2 + (y+3)^2} + y^2 + 6y + 9$$
$$16\sqrt{x^2 + (y+3)^2} = 64 + 12y$$
$$256[x^2 + (y+3)^2] = 4096 + 1536y + 144y^2$$
$$256[x^2 + y^2 + 6y + 9] = 4096 + 1536y + 144y^2$$
$$256x^2 + 256y^2 + 1536y + 2304 = 4096 + 1536y + 144y^2$$
$$256x^2 + 112y^2 = 1792$$
$$\frac{x^2}{7} + \frac{y^2}{16} = 1$$

Closing (2 minutes)

Give students a moment to respond to the questions below, and then call on them to share their responses with the whole class.

- Let F and G be the foci of an ellipse. If P and Q are points on the ellipse, what conclusion can you draw about distances from F and G to P and Q?
  - $PF + PG = QF + QG$
- What information do students need in order to derive the equation of an ellipse?
  - *The foci and the sum of the distances from the foci to a point on the ellipse.*
- What is fundamentally true about every ellipse?
  - *From every point on the ellipse, the sum of the distance to each focus is constant.*
- What is the standard form of an ellipse centered at the origin?
  - $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Exit Ticket (8 minutes)

Name ______________________________ Date ____________________

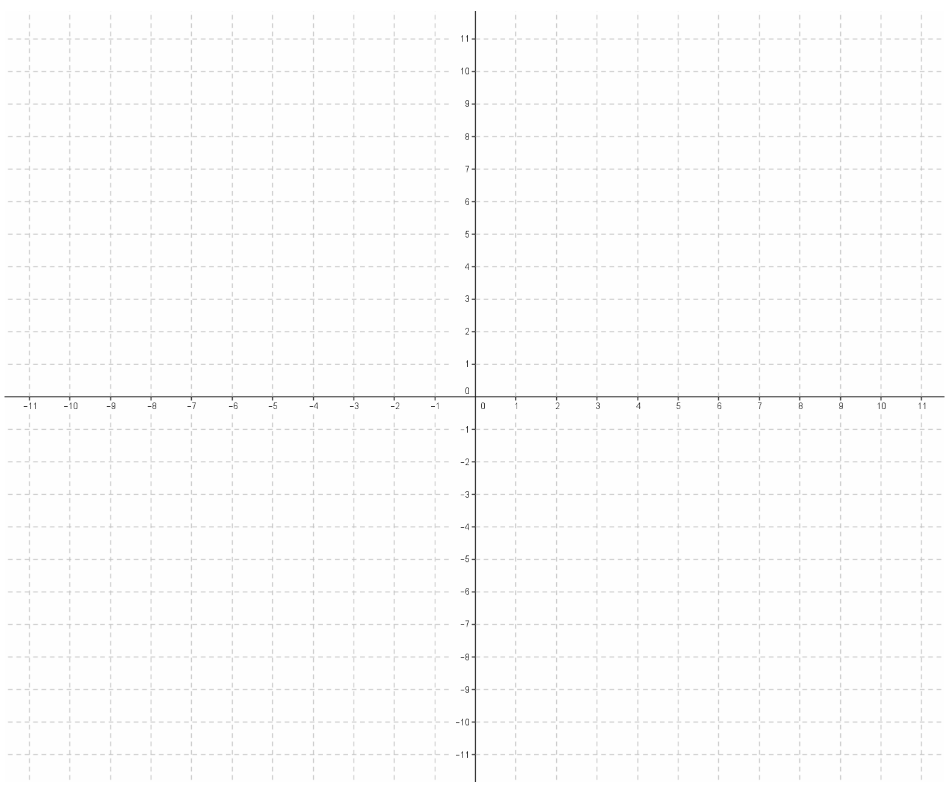
Lesson 7: Curves from Geometry

Exit Ticket

Suppose that the foci of an ellipse are $F(-1,0)$ and $G(1,0)$ and that the point $P(x,y)$ satisfies the condition $PF + PG = 4$.

a. Derive an equation of an ellipse with foci F and G that passes through P. Write your answer in standard form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

b. Sketch the graph of the ellipse defined above.



c. Verify that the x-intercepts of the graph satisfy the condition $PF + PG = 4$.

d. Verify that the y-intercepts of the graph satisfy the condition $PF + PG = 4$.

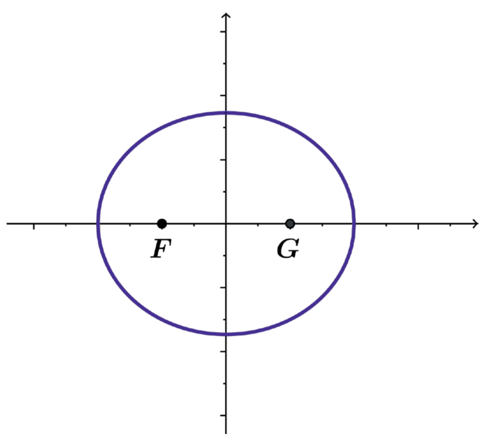
Exit Ticket Sample Solutions

Suppose that the foci of an ellipse are $F(-1,0)$ and $G(1,0)$ and that the point $P(x,y)$ satisfies the condition $PF+PG=4$.

a. **Derive an equation of an ellipse with foci F and G that passes through P. Write your answer in standard form: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.**

$$PF+PG=4$$
$$\sqrt{(x-1)^2+y^2}+\sqrt{(x+1)^2+y^2}=4$$
$$(x-1)^2+y^2=16-8\sqrt{(x+1)^2+y^2}+(x+1)^2+y^2$$
$$(x-1)^2-(x+1)^2-16=-8\sqrt{(x+1)^2+y^2}$$
$$x^2-2x+1-(x^2+2x+1)-16=-8\sqrt{(x+1)^2+y^2}$$
$$-4x-16=-8\sqrt{(x+1)^2+y^2}$$
$$x+4=2\sqrt{(x+1)^2+y^2}$$
$$x^2+8x+16=4(x^2+2x+1+y^2)$$
$$x^2+8x+16=4x^2+8x+4+4y^2$$
$$-3x^2-4y^2=-12$$
$$\frac{x^2}{4}+\frac{y^2}{3}=1$$

b. **Sketch the graph of the ellipse defined above.**



c. **Verify that the x-intercepts of the graph satisfy the condition $PF+PG=4$.**

For the x-intercepts $(2,0)$ and $(-2,0)$, we have $PF+PG=1+3=4$ and $PF+PG=3+1=4$.

d. **Verify that the y-intercepts of the graph satisfy the condition $PF+PG=4$.**

For the y-intercepts $(0,\sqrt{3})$ and $(0,-\sqrt{3})$, we have $\sqrt{\sqrt{3}^2+(-1)^2}+\sqrt{\sqrt{3}^2+1^2}=2+2=4$ and $\sqrt{\left(-\sqrt{3}\right)^2+(-1)^2}+\sqrt{\left(-\sqrt{3}\right)^2+1^2}=2+2=4$.

Problem Set Sample Solutions

1. Derive the equation of the ellipse with the given foci F and G that passes through point P. Write your answer in standard form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   a. The foci are $F(-2, 0)$ and $G(2, 0)$, and point $P(x, y)$ satisfies the condition $PF + PG = 5$.

   $$\frac{x^2}{\frac{25}{4}} + \frac{y^2}{\frac{9}{4}} = 1$$

   $$\frac{4x^2}{25} + \frac{4y^2}{9}$$

   b. The foci are $F(-1, 0)$ and $G(1, 0)$, and point $P(x, y)$ satisfies the condition $PF + PG = 5$.

   $$\frac{x^2}{\frac{25}{4}} + \frac{y^2}{\frac{21}{4}} = 1$$

   $$\frac{4x^2}{25} + \frac{4y^2}{21}$$

   c. The foci are $F(0, -1)$ and $G(0, 1)$, and point $P(x, y)$ satisfies the condition $PF + PG = 4$.

   $$\frac{x^2}{3} + \frac{y^2}{4} = 1$$

   d. The foci are $F\left(-\frac{2}{3}, 0\right)$ and $G\left(\frac{2}{3}, 0\right)$, and point $P(x, y)$ satisfies the condition $PF + PG = 3$.

   $$\frac{x^2}{\frac{9}{4}} + \frac{y^2}{\frac{65}{36}} = 1$$

   $$\frac{4x^2}{9} + \frac{36y^2}{65} = 1$$

   e. The foci are $F(0, -5)$ and $G(0, 5)$, and point $P(x, y)$ satisfies the condition $PF + PG = 12$.

   $$\frac{x^2}{11} + \frac{y^2}{36} = 1$$

   f. The foci are $F(-6, 0)$ and $G(6, 0)$, and point $P(x, y)$ satisfies the condition $PF + PG = 20$.

   $$\frac{x^2}{100} + \frac{y^2}{64} = 1$$

2. Recall from Lesson 6 that the semi-major axes of an ellipse are the segments from the center to the farthest vertices, and the semi-minor axes are the segments from the center to the closest vertices. For each of the ellipses in Problem 1, find the lengths a and b of the semi-major axes.

   a. $a = \frac{5}{2}, b = \frac{3}{2}$

   b. $a = \frac{5}{2}, b = \frac{\sqrt{21}}{2}$

   c. $a = \sqrt{3}, b = 2$

   d. $a = \frac{3}{2}, b = \frac{\sqrt{65}}{6}$

   e. $a = \sqrt{11}, b = 6$

   f. $a = 10, b = 8$

MP.8

3. Summarize what you know about equations of ellipses centered at the origin with vertices $(a, 0)$, $(-a, 0)$, $(0, b)$, and $(0, -b)$.

   For ellipses centered at the origin, the equation is always $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a is the positive x-value of the x-intercepts and b is the positive y-value of the y-intercepts. If we know the x- and y-intercepts, then we know the equation of the ellipse.

4. Use your answer to Problem 3 to find the equation of the ellipse for each of the situations below.

   a. An ellipse centered at the origin with x-intercepts $(-2, 0)$, $(2, 0)$ and y-intercepts $(0, 8)$, $(0, -8)$

   $$\frac{x^2}{4} + \frac{y^2}{64} = 1$$

   b. An ellipse centered at the origin with x-intercepts $(-\sqrt{5}, 0)$, $(\sqrt{5}, 0)$ and y-intercepts $(0, 3)$, $(0, -3)$

   $$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

5. Examine the ellipses and the equations of the ellipses you have worked with, and describe the ellipses with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the three cases $a > b$, $a = b$, and $b > a$.

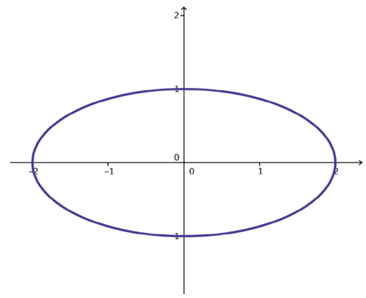
   If $a > b$, then the foci are on the x-axis and the ellipse is oriented horizontally, and if $b > a$, the foci are on the y-axis and the ellipse is oriented vertically. If $a = b$, then the ellipse is a circle with radius a.

6. Is it possible for $\frac{x^2}{4} + \frac{y^2}{9} = 1$ to have foci at $(-c, 0)$ and $(c, 0)$ for some real number c?

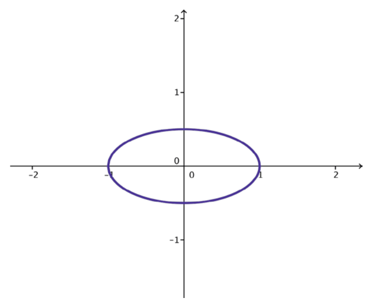
   No. Since $9 > 4$, the foci must be along the y-axis.

7. **For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation $\frac{x^2}{4} + y^2 = k$.**

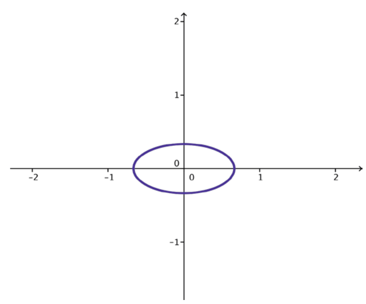
a. $k = 1$



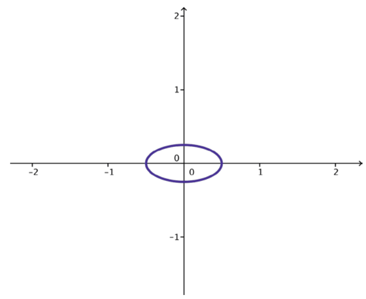
b. $k = \frac{1}{4}$



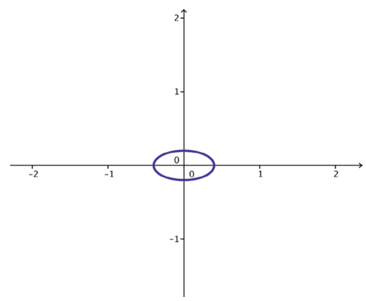
c. $k = \frac{1}{9}$



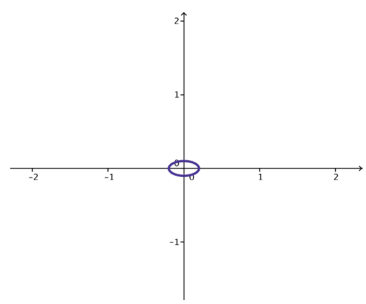
d. $k = \frac{1}{16}$



e. $k = \frac{1}{25}$



f. $k = \frac{1}{100}$



MP.3 & MP.7

g. **Make a conjecture: Which points in the plane satisfy the equation $\frac{x^2}{4} + y^2 = 0$?**

As k is getting smaller, the ellipse is shrinking. It seems that the only point that lies on the curve given by $\frac{x^2}{4} + y^2 = 0$ would be the single point $(0, 0)$.

h. **Explain why your conjecture in part (g) makes sense algebraically.**

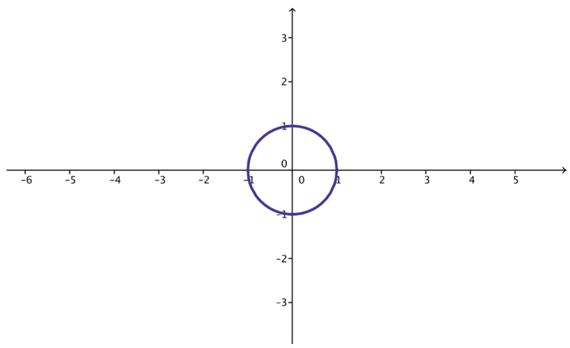
Both $\frac{x^2}{4}$ and y^2 are nonnegative numbers, and the only way to sum two nonnegative numbers and get zero would be if they were both zero. Thus, $\frac{x^2}{4} = 0$ and $y^2 = 0$, which means that (x, y) is $(0, 0)$.

i. **Which points in the plane satisfy the equation $\frac{x^2}{4} + y^2 = -1$?**

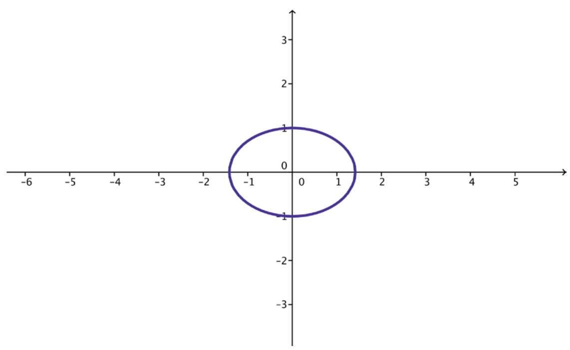
There are no points in the plane that satisfy the equation $x^2 + y^2 = -1$ because $\frac{x^2}{4} + y^2 \geq 0$ for all real numbers x and y.

8. **For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation $\frac{x^2}{k} + y^2 = 1$.**

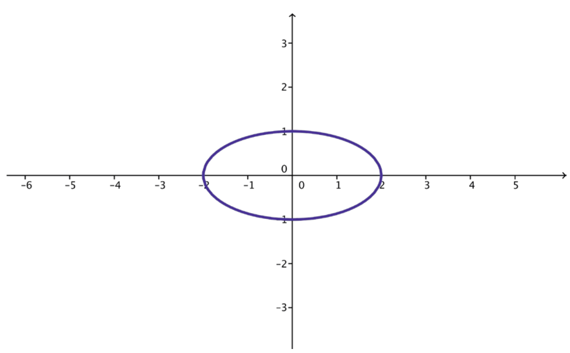
a. $k = 1$



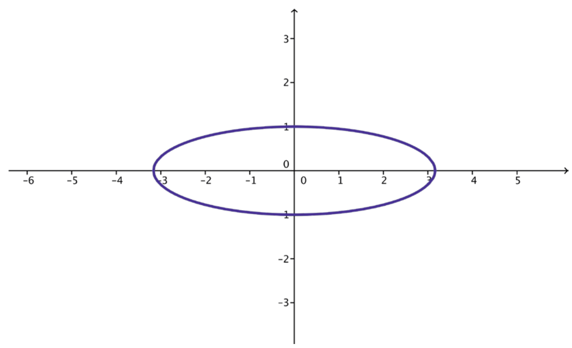
b. $k = 2$



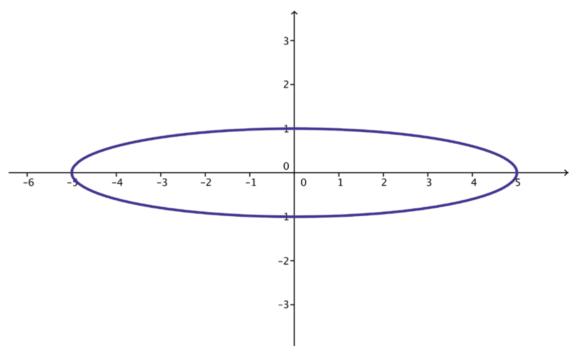
c. $k = 4$



d. $k = 10$



e. $k = 25$



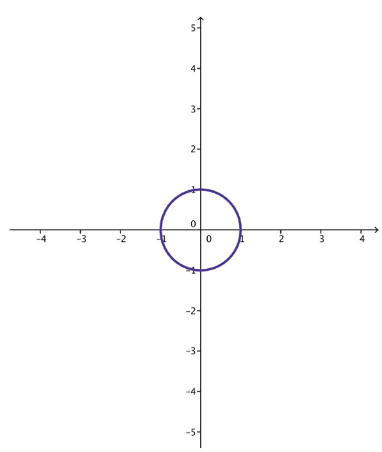
MP.7

f. Describe what happens to the graph of $\frac{x^2}{k} + y^2 = 1$ as $k \to \infty$.

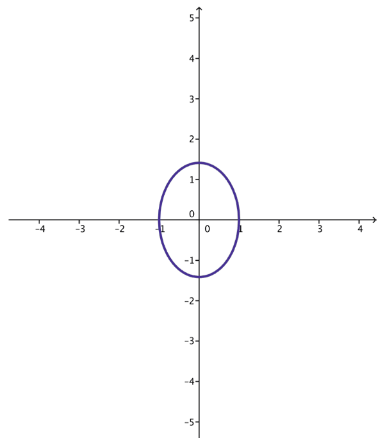
As k gets larger and larger, the ellipse stretches more and more horizontally, while not changing vertically. The vertices are $\left(-\sqrt{k}, 0\right)$, $\left(\sqrt{k}, 0\right)$, $(0, -1)$, and $(0, 1)$.

9. For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation $x^2 + \frac{y^2}{k} = 1$.

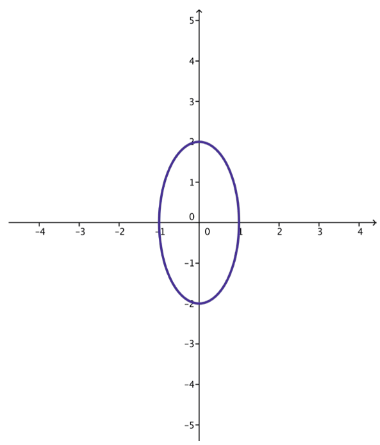
a. $k = 1$



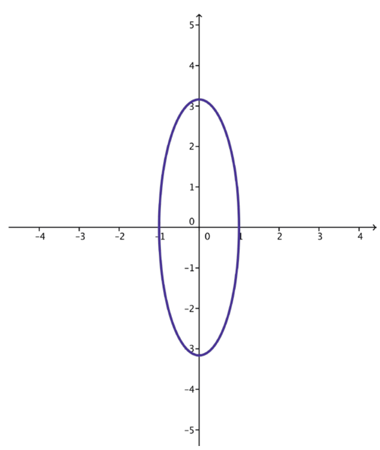
b. $k = 2$



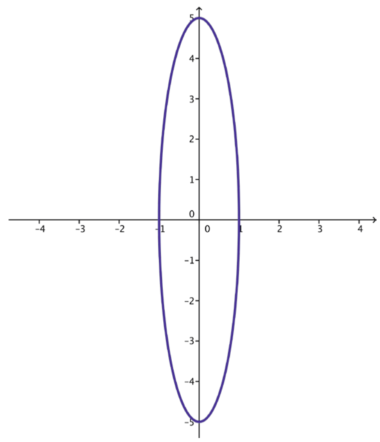
c. $k = 4$



d. $k = 10$



e. $k = 25$



MP.7

f. **Describe what happens to the graph of $x^2 + \frac{y^2}{k} = 1$ as $k \to \infty$.**

As k gets larger and larger, the ellipse stretches more and more vertically, while not changing horizontally. The vertices are $(-1, 0)$, $(1, 0)$, $\left(0, -\sqrt{k}\right)$, and $\left(0, \sqrt{k}\right)$.

Lesson 8: Curves from Geometry

Student Outcomes

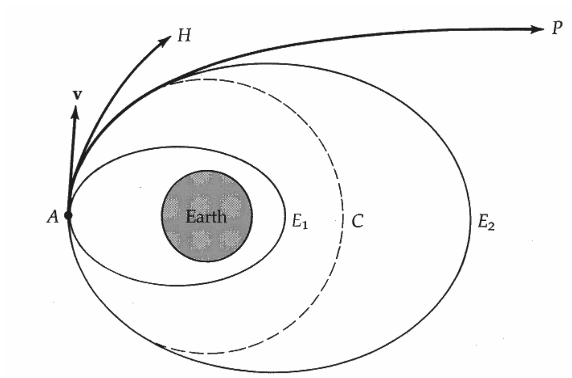
- Students learn to graph equations of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
- Students derive the equations of hyperbolas given the foci, using the fact that the difference of distances from the foci is constant (**G-GPE.A.3**).

Lesson Notes

In the previous lessons, students learned how to describe an ellipse, how to graph an ellipse, and how to derive the standard equation of an ellipse knowing its foci. In this lesson, students learn to perform these tasks for a hyperbola. The opening of the lesson establishes a connection between ellipses, parabolas, and hyperbolas in the context of the orbital path of a satellite.

Classwork

Opening (2 minutes)



Display this picture and the paragraph below for the class.

When a satellite moves in a closed orbit around a planet, it follows an elliptical path. However, if the satellite is moving fast enough, it overcomes the gravitational attraction of the planet and breaks out of its closed orbit. The minimum velocity required for a satellite to escape the closed orbit is called the escape velocity. The velocity of the satellite determines the shape of its orbit.

In pairs, have students answer the following questions. Call the class back together to debrief.

- If the velocity of the satellite is less than the escape velocity, it follows a path that looks like E_2 in the diagram. Describe the path, and give the mathematical term for this curve.
  - *Elliptical Path, Ellipse*

- If the velocity of the satellite is exactly equal to the escape velocity, it follows a path that looks like P in the diagram. Describe the path, and give the mathematical term for this curve.
  - *Parabolic Path, Parabola*
- If the velocity of the satellite exceeds the escape velocity, it follows a path that looks like H in the diagram. Describe the path, and give the mathematical term for this curve.

Students may describe the path in general terms but will not know the name. Tell them the name for this curve is a hyperbola.

All three trajectories are shown in the diagram. The ellipse and the parabola were studied in previous lessons; the focus of this lesson is the hyperbola.

Discussion (12 minutes): Analysis of $x^2 - y^2 = 1$

MP.2 & MP.3

- Consider the equation $x^2 + y^2 = 1$. When we graph the set of points (x, y) that satisfy this equation, what sort of curve do we get?
  - *The graph of this equation produces a circle. The center of the circle is $(0,0)$, and the radius of the circle is 1.*
- Let's make one small change to this equation: Consider $x^2 - y^2 = 1$.
- What sort of curve does this produce? Develop an argument, and share it with a neighbor. (Debrief as a class.)
- Let's explore this question together. We focus on three features that are of general interest when studying curves: intercepts, symmetries, and end behavior.
- Does this curve intersect the x-axis? Does it intersect the y-axis? If so, where?
  - *When $x = 0$, we have $-y^2 = 1$, which is equivalent to $y^2 = -1$. Since there are no real numbers that satisfy this equation, the graph of $x^2 - y^2 = 1$ does not intersect the y-axis.*
  - *When $y = 0$, we have $x^2 = 1$, which is true when $x = 1$ or $x = -1$. So the graph of $x^2 - y^2 = 1$ intersects the x-axis at $(1,0)$ and again at $(-1,0)$.*
- What sort of symmetries do you expect this graph to have?
  - *If (a, b) satisfies $x^2 - y^2 = 1$, then we have $a^2 - b^2 = 1$. We can see that $(-a, b)$ also satisfies the equation since $(-a)^2 - b^2 = a^2 - b^2 = 1$. Thus, the graph is symmetrical about the y-axis.*
  - *Similarly, the point $(a, -b)$ is on the graph whenever (a, b) is, showing us that the graph is symmetrical about the x-axis.*
- So if we can sketch a portion of the graph in the first quadrant, we can immediately infer what the rest of the graph looks like. That should come in handy.
- Now, let's try to get a feel for what this curve looks like. It never hurts to plot a few points, so let's start with that approach. Solving for y makes this process a bit easier, so go ahead and isolate the y-variable.
  - *We can write $y^2 = x^2 - 1$, which means $y = \pm\sqrt{x^2 - 1}$.*

Scaffolding:

- Some students may need guidance on how to determine the symmetries of the graph. Offer the following cues as needed: "If (a, b) satisfies the equation, does $(-a, b)$ also satisfy the equation? How can you tell? If both (a, b) and $(-a, b)$ satisfy the equation, what does this mean for the graph?"
- Ask advanced students to determine the key features of the curve produced by $x^2 - y^2 = 1$ independently, without scaffolded questions.

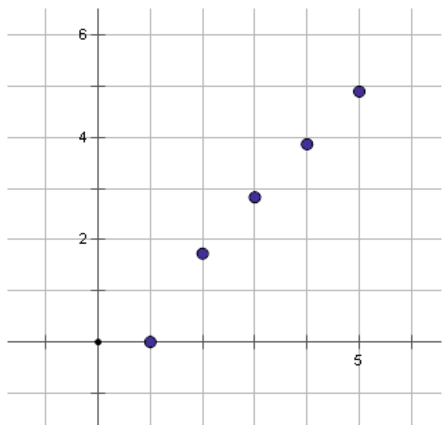
- Since we are only going to deal with points in the first quadrant, we can simply consider $y = \sqrt{x^2 - 1}$. Use this relation to find the y-values that correspond to $x = 1, 2, 3, 4$, and 5.

x	1	2	3	4	5
y	$\mathbf{0}$	$\boldsymbol{\sqrt{3}}$	$\boldsymbol{\sqrt{8}}$	$\boldsymbol{\sqrt{15}}$	$\boldsymbol{\sqrt{24}}$

- To get a feel for these numbers, use your calculator to obtain a decimal approximation for each square root.

x	1	2	3	4	5
y	$\mathbf{0}$	$\mathbf{1.73}$	$\mathbf{2.83}$	$\mathbf{3.87}$	$\mathbf{4.90}$

- Now, let's use a software tool to plot these points for us:

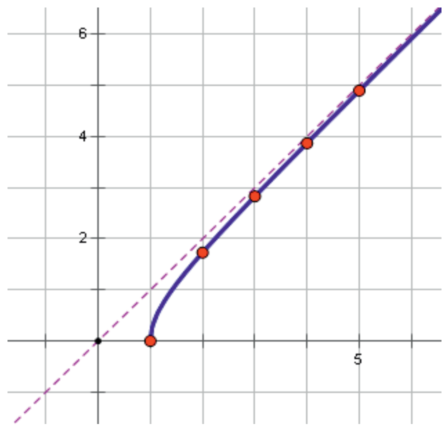


- Perhaps you are beginning to get a sense of what the graph looks like, but more data is surely useful. Let's use a spreadsheet or a graphing calculator to generate some more data. In particular, what happens to the y-values when the x-values get larger and larger? Let's have a look:

x	10	20	50	100	500
y	9.949	19.974	49.989	99.994	499.999

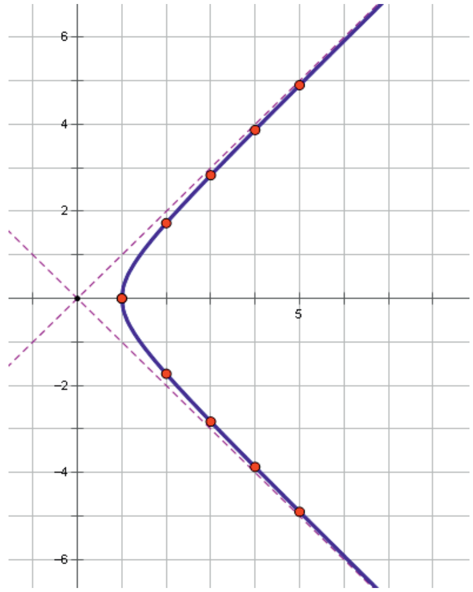
- Here the data are shown to three decimal places without rounding. Now this is interesting! Make some observations about the data in front of you, and then share those observations with a partner. In a minute, I will ask some of you to share your thinking with the class.
  - *The y-value is always smaller than the x-value.*
  - *As the value of x increases, the y-value seems to get closer and closer to the x-value.*
- So the data are telling us a particular story, but the power of mathematics lies in its ability to explain that story. Let's do some further analysis to see if we can confirm our conjectures, and possibly along the way we will discover why they are true.
- If (x, y) satisfies $x^2 - y^2 = 1$, is it true that $y < x$? How can we be sure?
  - *We have $y^2 = x^2 - 1$; that is, y^2 is 1 less than x^2. This means that $y^2 < x^2$, and since we're only considering points in the first quadrant, we must have $y < x$.*

- What does this tell us about the curve we are studying? In particular, where is the graph in relation to the line $y = x$?
  - *The relation $y < x$ tells us that the graph lies below the line $y = x$.*
- So apparently, the line $y = x$ acts as an upper boundary for the curve. Let's draw this boundary line for visual reference. We also use software to draw a portion of the curve for us:

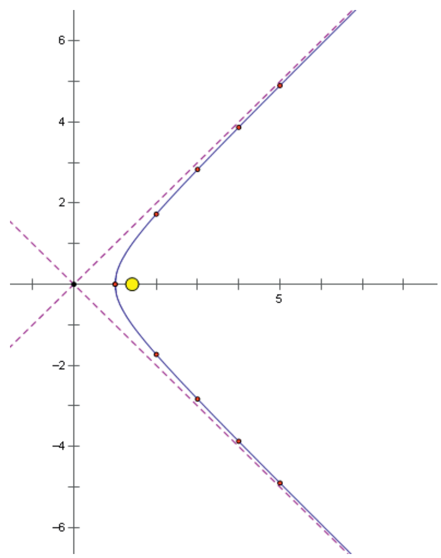


- Since the points on the curve are getting closer and closer to the line $y = x$, this graph provides visual confirmation of our observation that the y-value gets closer and closer to the x-value as x gets larger and larger. So now let's turn our attention to that conjecture: Why exactly must this be true? We approach this question a bit informally in this lesson.
- We have $y^2 = x^2 - 1$. If the value of x^2 is 100, then the value of y^2 must be 99. So when $x = \sqrt{100}$, $y = \sqrt{99}$.
- Let's get some practice with this kind of thinking. Do the following exercises with a partner. Partner A, complete the sentence, "If the value of x^2 is $1{,}000$, then …." Partner B, complete the sentence, "If the value of x^2 is 1,000,000, then …."
  - *We have $y^2 = x^2 - 1$. If the value of x^2 is $1{,}000$, then the value of y^2 must be 999. So when $x = \sqrt{1000}$, $y = \sqrt{999}$.*
  - *We have $y^2 = x^2 - 1$. If the value of x^2 is $1{,}000{,}000$, then the value of y^2 must be $999{,}999$. So when $x = \sqrt{1\,000\,000}$, $y = \sqrt{999\,999}$.*
- Even without picking up a calculator, does it make sense to you intuitively that the value of $\sqrt{999\,999}$ is extremely close to $\sqrt{1\,000\,000}$? When we check this on a calculator, we see that the values are 999.9995 and $1{,}000.0000$, respectively. Those values are indeed close.
- To sum up, we have four important facts about the graph of $x^2 - y^2 = 1$: First, the graph contains the point $(1,0)$. Second, the graph lies below the line $y = x$. Third, as we move along to the right, the points on the curve get extremely close to the line $y = x$. And fourth, the graph is symmetric with respect to both the x-axis and the y-axis.

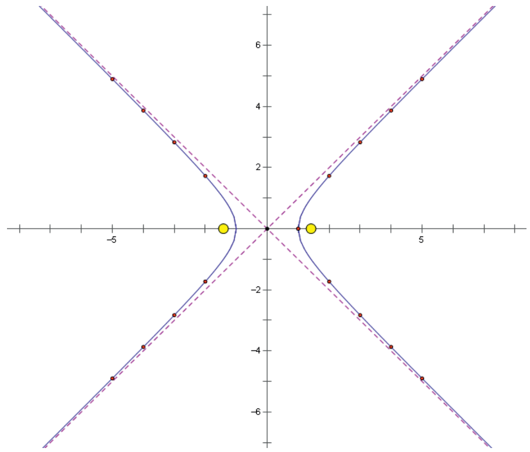
- Let's bring symmetry to bear on this discussion. Since the graph is symmetric with respect to the x-axis, we can infer the location of points in the fourth quadrant:



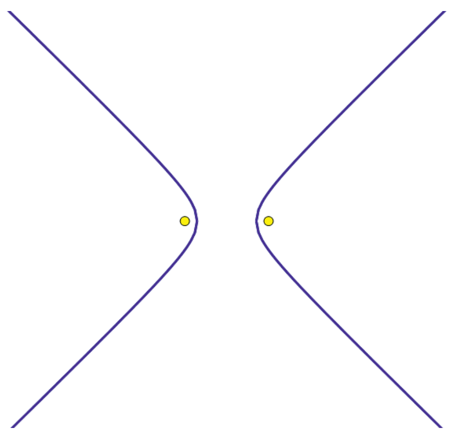
- Does this remind you of anything from the start of today's lesson? That's right! It's the trajectory followed by a satellite that has exceeded its escape velocity. This curve is called a *hyperbola*; therefore, we say that the satellite is following a *hyperbolic path.*



- In the previous above, the yellow dot could represent a large body, such as a planet or the sun, and the blue curve could represent the trajectory of a *hyperbolic comet,* which is a comet moving at such great speed that it follows a hyperbolic path. The boundary lines, which for this hyperbola have equations $y = x$ and $y = -x$, are known as *oblique* or *slant asymptotes*. The location of the yellow dot is called a *focus* of the hyperbola. In fact, since the curve is also symmetrical with respect to the y-axis, a hyperbola actually has two foci:



- So that you can be perfectly clear on what a hyperbola really looks like, here is an image that contains just the curve and the two foci:



- Turn to your neighbor, and answer the original question: What sort of curve results when we graph the set of points that satisfy $x^2 - y^2 = 1$? Describe this curve in as much detail as you can.
    - *The graph of $x^2 - y^2 = 1$ is a hyperbola that has intercepts on the x-axis and gets very close to the asymptotes $y = x$ and $y = -x$.*

Discussion (6 minutes): Analysis of $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$

- Now that we have a sense for what a hyperbola looks like, let's analyze the basic equation. We began with $x^2 - y^2 = 1$.
- What do you suppose would happen if we took the curve $x^2 - y^2 = 1$ and replaced x and y with $\frac{x}{3}$ and $\frac{y}{2}$?

Allow the students to have a few minutes to think about the answer to this question and discuss it quietly with a partner before continuing.

- Rewrite the equation $\left(\frac{x}{3}\right)^2 - \left(\frac{y}{2}\right)^2 = 1$ without parentheses.
  - $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$
- Find the x-intercepts.
  - $$\frac{x^2}{3^2} - \frac{0^2}{2^2} = 1$$
    $$\frac{x^2}{3^2} = 1$$
    $$x^2 = 9$$
    $$x = \pm 3$$
    The x-intercepts are $(3, 0)$ and $(-3, 0)$.
- How do the x-intercepts compare to those of the graph of $x^2 - y^2 = 1$?
  - *The x-intercepts are now $(3,0)$ and $(-3,0)$ instead of $(1, 0)$ and $(-1, 0)$.*

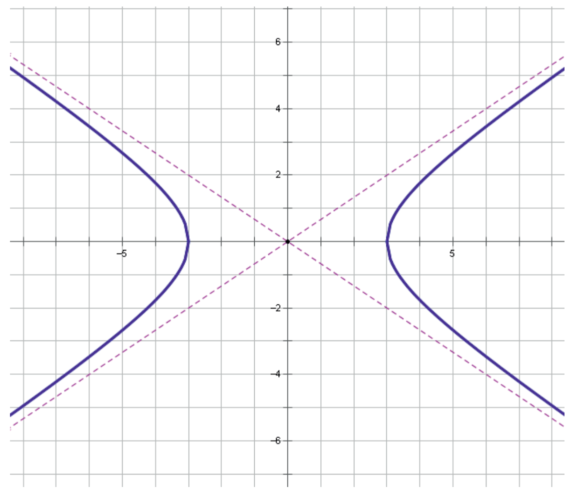
MP.2 & MP 3

- Do you think the graph would still be symmetrical?
  - *Yes, the graph should still be symmetrical for exactly the same reasons as before.*

If graphing software is available, show the graph of both equations on the same axis. If software is not available, plot points to construct each graph.

- What can we say about the end behavior of this curve? Let's think about this together. In the equation $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$, suppose that the value of the first term was $100{,}000$. What value would be needed for the second term?
  - *To satisfy the equation, the value of the second term would have to be* $99{,}999$.
- So we need to have $\frac{x}{3} = \sqrt{100\,000} \approx 316.227$ and $\frac{y}{2} = \sqrt{99999} \approx 316.226$. Whereas before we had $y \approx x$ for large values of x, notice that in this case $\frac{y}{2} \approx \frac{x}{3}$. Thus, if x is a large number, then $\frac{y}{2} \approx \frac{x}{3}$, and so $y \approx \frac{2}{3}x$. The larger x becomes, the better this approximation becomes.
- Let's summarize this part of the discussion. What was the boundary line for points on the curve generated by the original equation $x^2 - y^2 = 1$? What is the boundary line for points on the new curve $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$?
  - *For $x^2 - y^2 = 1$, the line $y = x$ was an asymptote for the curve.*
  - *For $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$, the line $y = \frac{2}{3}x$ is an asymptote for the curve.*

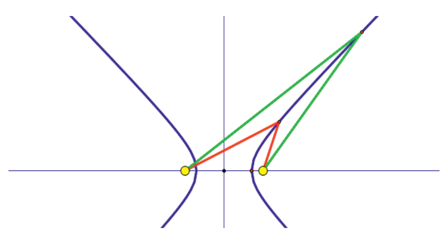
- With these things in mind, can you sketch the graph of the equation $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$?



- Notice that this is really the same shape as before; it is just that the plane has been stretched horizontally by a factor of 3 and vertically by a factor of 2.
- Where do you suppose the foci of this curve are located? This is not an easy question. Recall that in the last lesson, we used the foci of an ellipse to generate an equation for the ellipse. We can use an analogous procedure to find the relationship between the foci of a hyperbola and the equation that generates the hyperbola.

Discussion (2 minutes): Formal Properties of Hyperbolas

- Like the ellipse, the formal definition of the hyperbola involves distances. In the figure below, the red segments shown differ in length by a certain amount; the green segments differ in length by exactly the same amount.

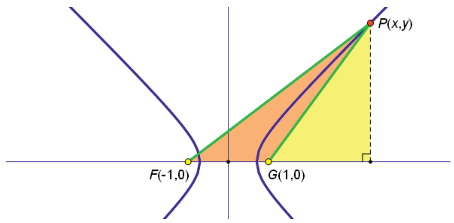


- This suggests the following formal definition: Given two points F and G, a *hyperbola* is a set of points P such that the difference $PF - PG$ is constant; that is, there is some number k such that $PF - PG = k$. The points F and G are called the *foci* of the hyperbola.
- As we show in the upcoming example, a hyperbola can be described by an equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. This is called the *standard equation* of the hyperbola. Note that some hyperbolas are described by an equation with this form: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. (Which kinds of curves go with which equation? We explore this in a moment.)

Example (7 minutes)

In this example, students use the foci of a hyperbola to derive an equation for the hyperbola.

- Let's take $F(-1, 0)$ and $G(1, 0)$ to be the foci of a hyperbola, with each point P on the hyperbola satisfying either $PF - PG = 1$ or $PG - PF = 1$. What is the equation of such a hyperbola? In particular, can we express the equation in the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$? The diagram below may help you to get started.



  - *We can use the distance formula to see that $PF = \sqrt{(x+1)^2 + y^2}$ and $PG = \sqrt{(x-1)^2 + y^2}$. Thus, we need to have $PF - PG = \sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2} = 1$.*

- In the last lesson, we learned a technique that can be used to deal with equations that contain two radical expressions. Apply that technique to this equation, and see where you end up!

  - $$\begin{aligned}
\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2} &= 1 \\
\sqrt{(x+1)^2 + y^2} &= 1 + \sqrt{(x-1)^2 + y^2} \\
(x+1)^2 + y^2 &= 1 + 2\sqrt{(x-1)^2 + y^2} + (x-1)^2 + y^2 \\
x^2 + 2x + 1 &= 1 + 2\sqrt{(x-1)^2 + y^2} + x^2 - 2x + 1 \\
4x - 1 &= 2\sqrt{(x-1)^2 + y^2} \\
16x^2 - 8x + 1 &= 4[(x-1)^2 + y^2] \\
16x^2 - 8x + 1 &= 4[x^2 - 2x + 1 + y^2] \\
16x^2 - 8x + 1 &= 4x^2 - 8x + 4 + 4y^2 \\
12x^2 - 4y^2 &= 3 \\
4x^2 - \frac{4}{3}y^2 &= 1
\end{aligned}$$

- This equation looks much friendlier than the one we started with. Rewrite the equation in the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

  - $\frac{x^2}{\frac{1}{4}} - \frac{y^2}{\frac{3}{4}} = 1$

- Thus, we have $a^2 = \frac{1}{4}$ and $b^2 = \frac{3}{4}$ so that $a = \sqrt{\frac{1}{4}}$ and $b = \sqrt{\frac{3}{4}}$.

Exercises (8 minutes)

Give students time to work on the following exercises. Monitor their work, and give assistance to individual students as needed. Encourage students to work in pairs.

Exercises

1. **Let $F(0,5)$ and $G(0,-5)$ be the foci of a hyperbola. Let the points $P(x,y)$ on the hyperbola satisfy either $PF - PG = 6$ or $PG - PF = 6$. Use the distance formula to derive an equation for this hyperbola, writing your answer in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.**

$$PG - PF = 6$$
$$\sqrt{x^2 + (y+5)^2} - \sqrt{x^2 + (y-5)^2} = 6$$
$$\sqrt{x^2 + (y+5)^2} = 6 + \sqrt{x^2 + (y-5)^2}$$
$$x^2 + (y^2 + 10y + 25) = 36 + 12\sqrt{x^2 + (y-5)^2} + (x^2 + y^2 - 10y + 25)$$
$$20y - 36 = 12\sqrt{x^2 + (y-5)^2}$$
$$5y - 9 = 3\sqrt{x^2 + (y-5)^2}$$
$$25y^2 - 90y + 81 = 9(x^2 + y^2 - 10y + 25)$$
$$25y^2 - 90y + 81 = 9x^2 + 9y^2 - 90y + 225$$
$$16y^2 - 9x^2 = 144$$
$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

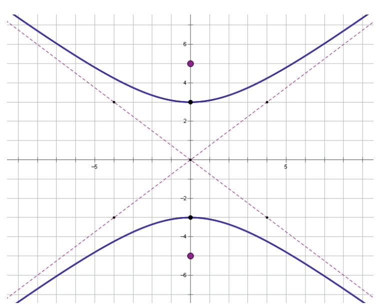
2. **Where does the hyperbola described above intersect the y-axis?**

The curve intersects the y-axis at $(0,3)$ and $(0,-3)$.

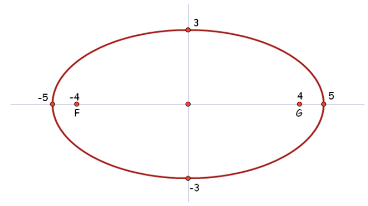
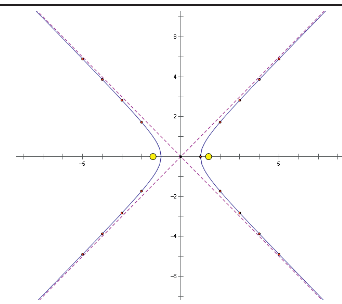
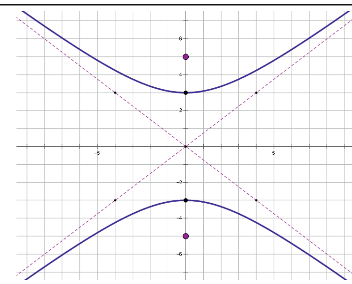
3. **Find an equation for the line that acts as a boundary for the portion of the curve that lies in the first quadrant.**

For large values of x, $\frac{y}{3} \approx \frac{x}{4}$, so the line $y = \frac{3}{4}x$ is the boundary for the curve in the first quadrant.

4. **Sketch the graph of the hyperbola described above.**



Complete the graphic organizer comparing ellipses and hyperbolas. A blank copy is attached at the end of the lesson.

Curve	Equation	Center	Asymptotes	Symmetry	Graph
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$(0,0)$	None	About center	
Hyperbola opening up/down	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$(0,0)$	$y=\frac{a}{b}x$ $y=-\frac{a}{b}x$	About y-axis	
Hyperbola opening left/right	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$	$(0,0)$	$y=\frac{a}{b}x$ $y=-\frac{a}{b}x$	About x-axis	

Closing (3 minutes)

Instruct students to write responses to the questions below in their notebooks. Call on students to share their responses with the class.

- What is the definition of a hyperbola? How is this definition different from that of the ellipse?
  - *Given two points F and G, a hyperbola is a set of points P such that the difference $PF - PG$ is constant.*
  - *For the ellipse, the sum $PF + PG$ is constant (rather than the difference).*
- What is the standard equation of a hyperbola? How is this equation different from that of the ellipse?
  - *The standard equation is either* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *or* $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.
  - *For the ellipse, the standard equation is* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- What are the equations of the asymptotes for the graph of the equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$?
  - *The equations for the asymptotes are* $y = \frac{a}{b}x$ *and* $y = -\frac{a}{b}x$.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 8: Curves from Geometry

Exit Ticket

Let $F(-4,0)$ and $B(4,0)$ be the foci of a hyperbola. Let the points $P(x, y)$ on the hyperbola satisfy either $PF - PG = 4$ or $PG - PF = 4$. Derive an equation for this hyperbola, writing your answer in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Exit Ticket Sample Solutions

Let $F(-4,0)$ and $B(4,0)$ be the foci of a hyperbola. Let the points $P(x,y)$ on the hyperbola satisfy either $PF - PG = 4$ or $PG - PF = 4$. Derive an equation for this hyperbola, writing your answer in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$$PF = \sqrt{(x+4)^2 + y^2}$$
$$PB = \sqrt{(x-4)^2 + y^2}$$
$$\sqrt{(x+4)^2 + y^2} = 4 + \sqrt{(x-4)^2 + y^2}$$
$$(x+4)^2 + y^2 = 16 + 8\sqrt{(x-4)^2 + y^2} + (x-4)^2 + y^2$$
$$x^2 + 8x + 16 + y^2 = 16 + 8\sqrt{(x-4)^2 + y^2} + x^2 - 8x + 16 + y^2$$
$$16x - 16 = 8\sqrt{(x-4)^2 + y^2}$$
$$2x - 2 = \sqrt{(x-4)^2 + y^2}$$
$$4x^2 - 8x + 4 = (x-4)^2 + y^2$$
$$4x^2 - 8x + 4 = x^2 - 8x + 16 + y^2$$
$$3x^2 - y^2 = 12$$
$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

Problem Set Sample Solutions

1. For each hyperbola described below: (1) Derive an equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. (2) State any x- or y-intercepts. (3) Find the equations for the asymptotes of the hyperbola.

   a. Let the foci be $A(-2,0)$ and $B(2,0)$, and let P be a point for which either $PA - PB = 2$ or $PB - PA = 2$.

      i. $x^2 - \frac{y^2}{3} = 1$

      ii. $(-1,0)$, $(1,0)$; *no y-intercepts*

      iii. $y \approx \sqrt{3}x$, *so* $y = \pm\sqrt{3}x$

   b. Let the foci be $A(-5,0)$ and $B(5,0)$, and let P be a point for which either $PA - PB = 5$ or $PB - PA = 5$.

      i. $\frac{x^2}{\frac{25}{4}} - \frac{y^2}{\frac{75}{4}} = 1$

      ii. $(-2.5,0)$, $(2.5,0)$; *no y-intercepts*

      iii. $y \approx \frac{\sqrt{75}}{2} \cdot \frac{2}{5}x = \sqrt{3}x$, *so* $y = \pm\sqrt{3}x$

   c. Consider $A(0,-3)$ and $B(0,3)$, and let P be a point for which either $PA - PB = 2.5$ or $PB - PA = 2.5$.

      i. $\frac{y^2}{\frac{9}{4}} - \frac{x^2}{\frac{27}{4}} = 1$

      ii. $(0,1.5)$, $(0,-1.5)$; *no x-intercepts*

      iii. $y \approx \frac{3}{2} \cdot \frac{2}{\sqrt{27}}x = \frac{1}{\sqrt{3}}x$, *so* $y = \pm\frac{\sqrt{3}}{3}x$

d. **Consider $A(0, -\sqrt{2})$ and $B(0, \sqrt{2})$, and let P be a point for which either $PA - PB = 2$ or $PB - PA = 2$.**

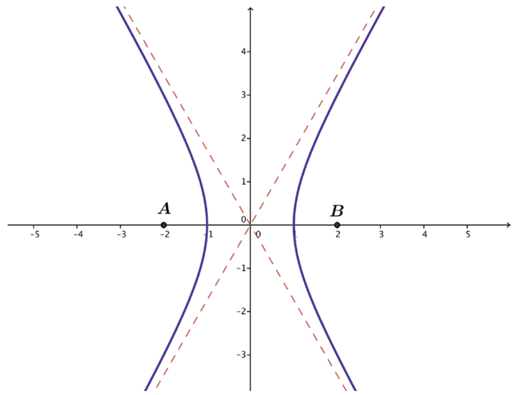
i. $y^2 - x^2 = 1$

ii. $(0, 1)$, $(0, -1)$; *no x-intercepts.*

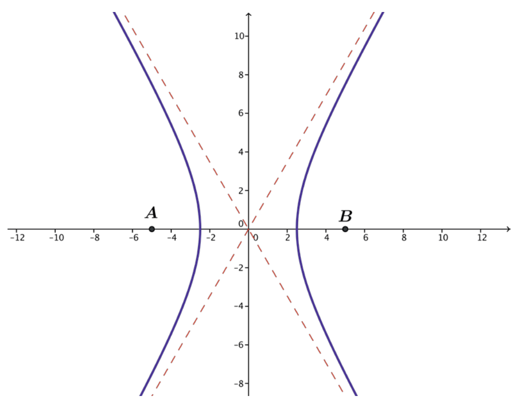
iii. $y = \pm x$

2. Graph the hyperbolas in parts (a)–(d) in Problem 1.

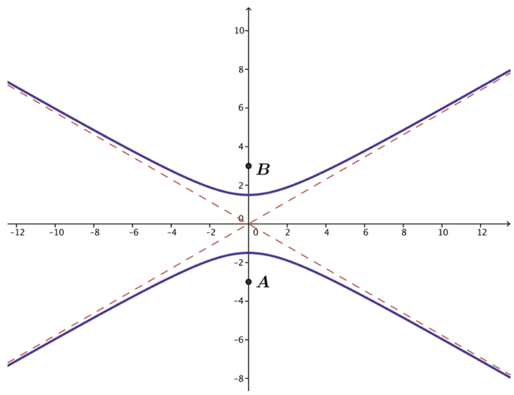
a.



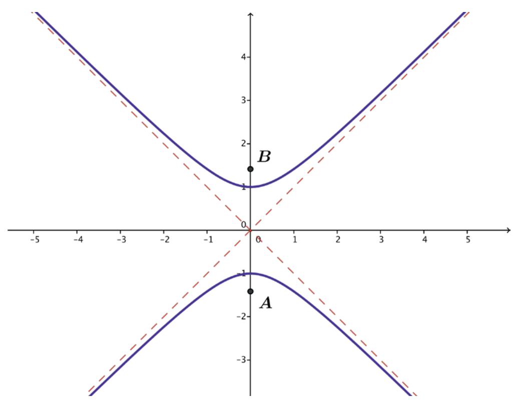
b.



c.

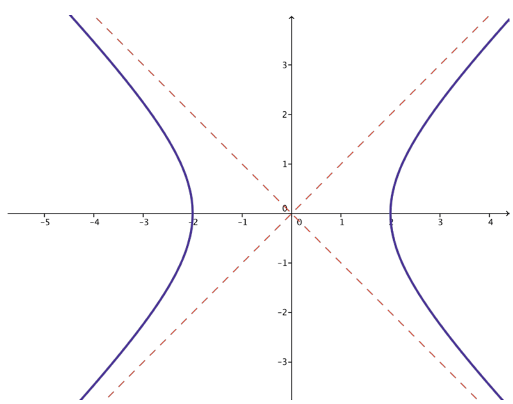


d.

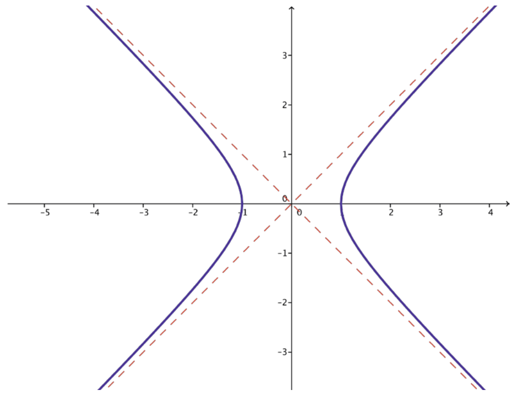


2. **For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation $x^2 - y^2 = k$.**

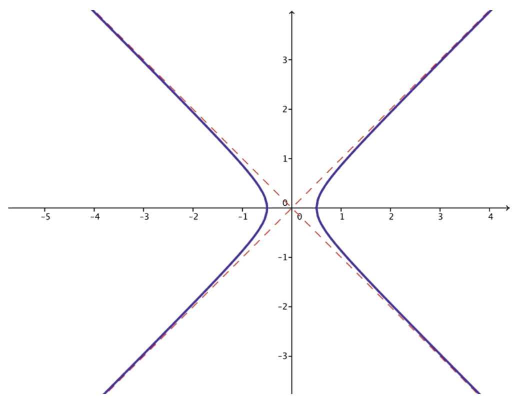
   a. $k = 4$



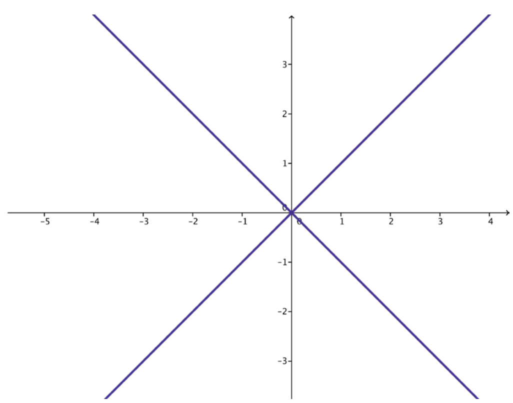
b. $k = 1$



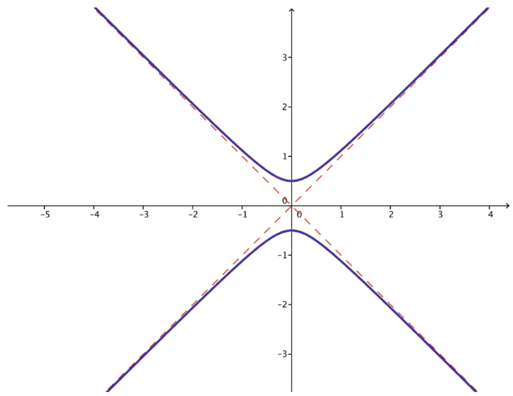
c. $k = \frac{1}{4}$



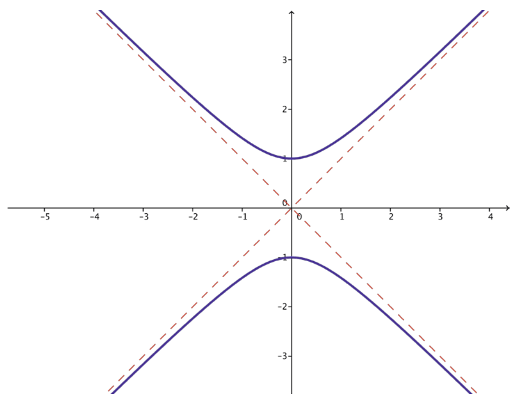
d. $k = 0$



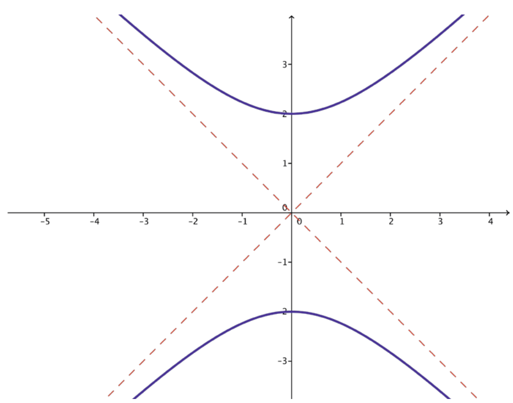
e. $k = -\frac{1}{4}$



f. $k = -1$



g. $k = -4$



MP.7

h. Describe the hyperbolas $x^2 - y^2 = k$ for different values of k. Consider both positive and negative values of k, and consider values of k close to zero and far from zero.

If k is close to zero, then the hyperbola is very close to the asymptotes $y = x$ and $y = -x$, appearing almost to have corners as the graph crosses the x-axis. If k is far from zero, the hyperbola gets farther from the asymptotes near the center. If $k > 0$, then the hyperbola crosses the x-axis, opening to the right and left, and if $k < 0$, then the hyperbola crosses the y-axis, opening up and down.

i. Are there any values of k so that the equation $x^2 - y^2 = k$ has no solution?

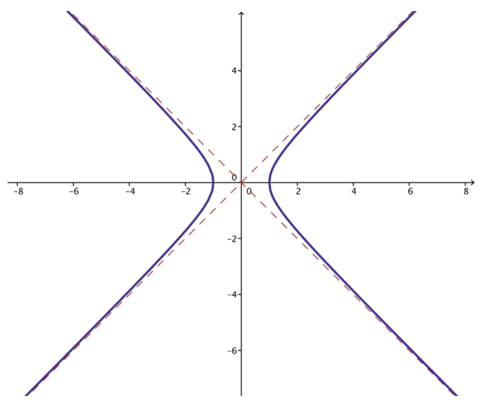
No. The equation $x^2 - y^2 = k$ always has solutions. The solution points lie on either two intersecting lines or on a hyperbola.

4. For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation $\frac{x^2}{k} - y^2 = 1$.

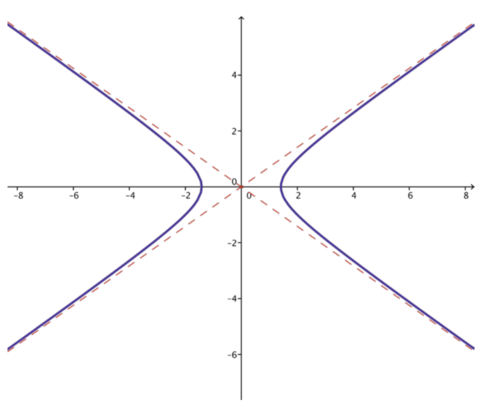
a. $k = -1$

There is no solution to the equation $\frac{x^2}{-1} - y^2 = 1$, because there are no real numbers x and y so that $x^2 + y^2 = -1$.

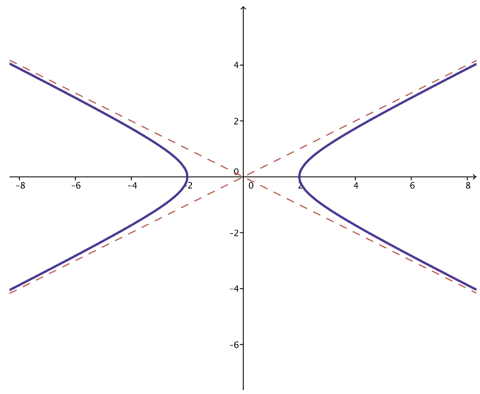
b. $k = 1$



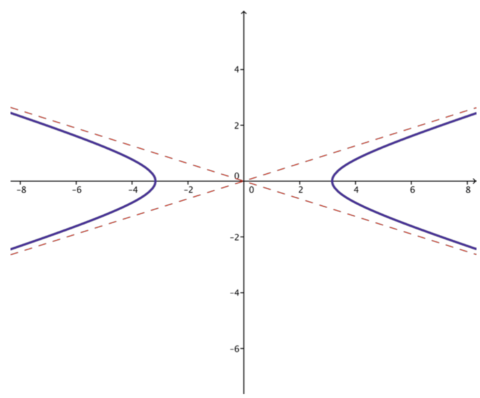
c. $k = 2$



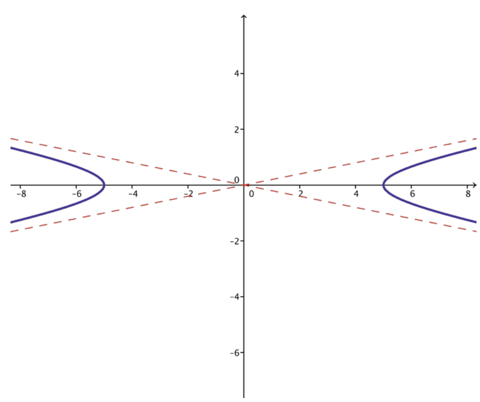
d. $k = 4$



e. $k = 10$



f. $k = 25$



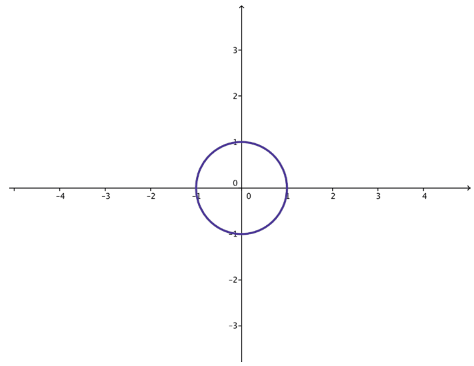
MP.7

g. **Describe what happens to the graph of $\frac{x^2}{k} - y^2 = 1$ as $k \to \infty$.**

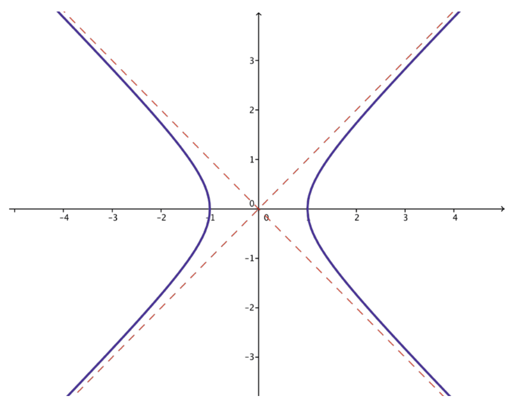
As $k \to \infty$, it appears that the hyperbolas with equation $\frac{x^2}{k} - y^2 = 1$ get flatter; the x-intercepts get farther from the center at the origin, and the asymptotes get less steep.

5. For each value of k specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation $x^2 - \frac{y^2}{k} = 1$.

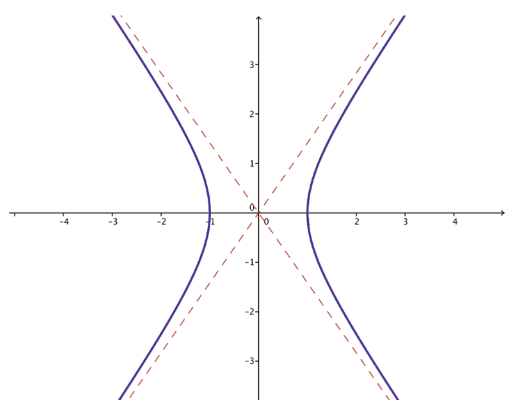
a. $k = -1$



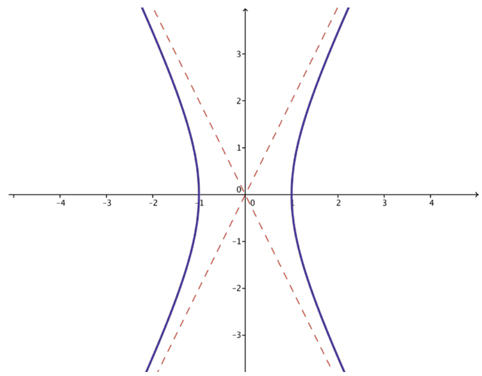
b. $k = 1$



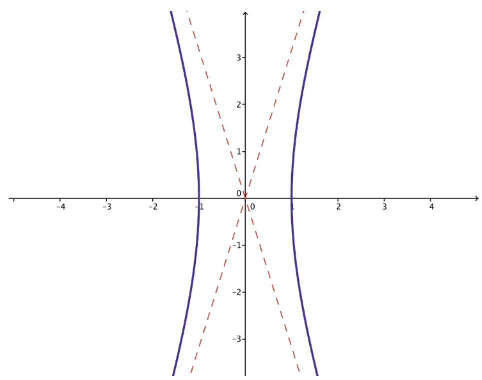
c. $k = 2$



d. $k = 4$



e. $k = 10$



MP.7

f. Describe what happens to the graph $x^2 - \frac{y^2}{k} = 1$ as $k \to \infty$.

As $k \to \infty$, the hyperbola with equation $x^2 - \frac{y^2}{k} = 1$ is increasingly stretched vertically. The center and intercepts do not change, but the steepness of the asymptotes increases.

6. An equation of the form $ax^2 + bx + cy^2 + dy + e = 0$ where a and c have opposite signs might represent a hyperbola.

a. Apply the process of completing the square in both x and y to convert the equation $9x^2 - 36x - 4y^2 - 8y - 4 = 0$ to one of the standard forms for a hyperbola: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$.

$$9x^2 - 36x - 4y^2 - 8y = 4$$
$$9(x^2 - 4x) - 4(y^2 + 2y) = 4$$
$$9(x^2 - 4x + 4) - 4(y^2 + 2y + 1) = 4 + 36 - 4$$
$$9(x - 2)^2 - 4(y + 1)^2 = 36$$
$$\frac{(x-2)^2}{4} - \frac{(y+1)^2}{9} = 1$$

b. Find the center of this hyperbola.

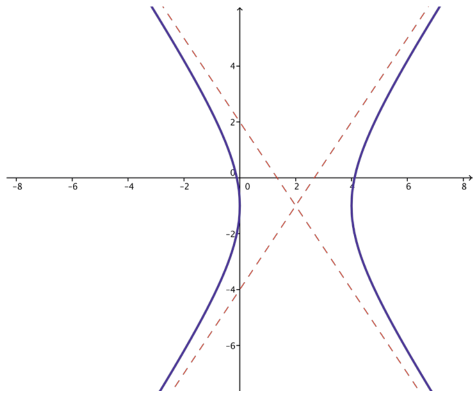
The center is $(2, -1)$.

c. **Find the asymptotes of this hyperbola.**

$\frac{x-2}{2} = \pm\frac{y+1}{3}$

$y = \frac{3}{2}x - 4$ *or* $y = -\frac{3}{2}x + 2$

d. **Graph the hyperbola.**



7. **For each equation below, identify the graph as either an ellipse, a hyperbola, two lines, or a single point. If possible, write the equation in the standard form for either an ellipse or a hyperbola.**

a. $4x^2 - 8x + 25y^2 - 100y + 4 = 0$

In standard form, this is the equation of an ellipse: $\frac{(x-1)^2}{25} + \frac{(y-2)^2}{4} = 1.$

b. $4x^2 - 16x - 9y^2 - 54y - 65 = 0$

When we try to put this equation in standard form, we find $\frac{(x-2)^2}{9} - \frac{(y+3)^2}{4} = 0$*, which gives* $\frac{x-2}{3} = \pm\frac{y+3}{2}$*. These are the lines with equation* $y = \frac{1}{3}(2x - 13)$ *and* $y = \frac{1}{3}(-2x - 5)$*.*

c. $4x^2 + 8x + y^2 + 2y + 5 = 0$

When we try to put this equation in standard form, we find $(x+1)^2 + \frac{(y+1)^2}{4} = 0$*. The graph of this equation is the single point* $(-1, -1)$*.*

d. $-49x^2 + 98x + 4y^2 - 245 = 0$

In standard form, this is the equation of a hyperbola: $\frac{y^2}{49} - \frac{(x-1)^2}{4} = 1.$

MP.7

e. **What can you tell about a graph of an equation of the form $ax^2 + bx + cy^2 + dy + e = 0$ by looking at the coefficients?**

There are two categories; if the coefficients a and c have the same sign, then the graph is either an ellipse, a point, or an empty set. If the coefficients a and c have opposite signs, then the graph is a hyperbola or two intersecting lines. We cannot tell just by looking at the coefficients which of these sub-cases hold.

Curve	Equation	Center	Asymptotes	Symmetry	Graph
Ellipse					
Hyperbola opening up/down					
Hyperbola opening left/right					

Lesson 9: Volume and Cavalieri's Principle

Student Outcomes

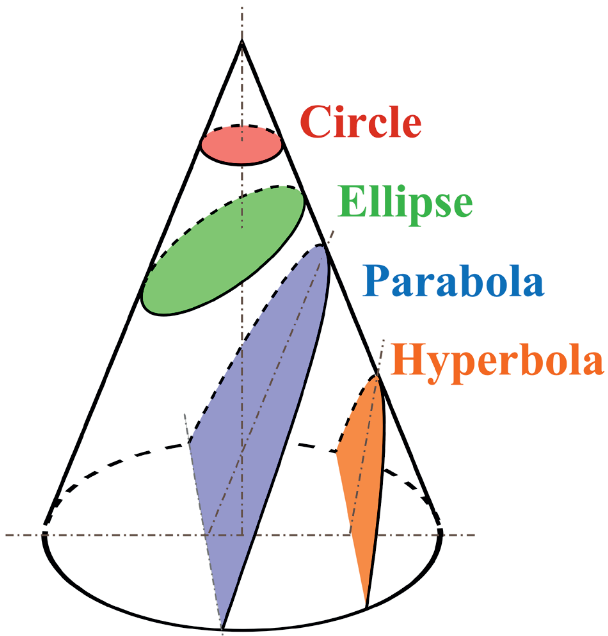
- Students are able to give an informal argument using Cavalieri's principle for the formula for the volume of a sphere and other solid figures (**G-GMD.A.2**).

Lesson Notes

The Opening uses the idea of cross sections to establish a connection between the current lesson and the previous lessons. In particular, ellipses and hyperbolas are seen as cross sections of a cone, and Cavalieri's volume principle is based on cross-sectional areas. This principle is used to explore the volume of pyramids, cones, and spheres.

Classwork

Opening (2 minutes)



"Conic Sections" by Magister Mathematicae is licensed under CC BY-SA 3.0
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Scaffolding:
- A cutout of a cone is available in Geometry Module 3 Lesson 7 to make picturing this exercise easier.
- Use the cutout to model determining that a circle is a possible cross section of a cone.

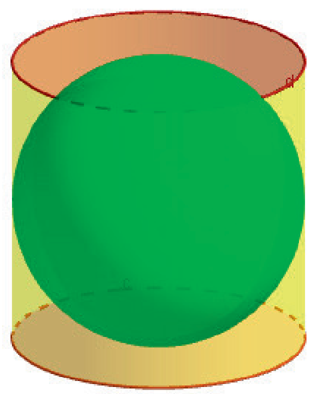
In the previous lesson, we saw how the ellipse, parabola, and hyperbola can come together in the context of a satellite orbiting a body such as a planet; we learned that the velocity of the satellite determines the shape of its orbit. Another context in which these curves and a circle arise is in slicing a cone. The intersection of a plane with a solid is called a *cross section* of the solid.

- Imagine a cone. How many different cross sections could you make by slicing the cone from any angle? Make a sketch of each one, and then share your results with a neighbor.
  - *Cross sections could be parabolas, circles, ellipses, and hyperbolas. A point would also occur if you slice right at the vertex of the cone, but this would not actually be a slice.*

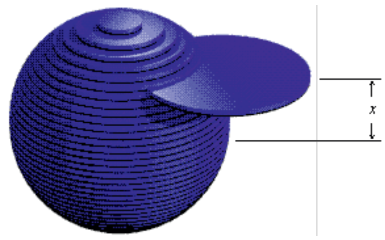
The figure above shows that these four curves are cross sections of the cone, which explains why they are often referred to as the *conic sections.*

In this lesson, we use cross sections to discover a relationship between cones, cylinders, and spheres. In particular, we derive the formula for the volume of a sphere. This formula was used to solve problems in Geometry, so today we focus on the derivation of that formula.

Discussion (4 minutes): A Sphere Enclosed by a Cylinder



- Imagine that a spherical balloon filled with water is placed into a cylindrical container as shown above. If you took a pin and pricked the balloon, allowing the water to leak out into the cylinder, how high would the water go? Would the water fill more or less than 50% of the cylinder? More or less than 90%? Write down your best guess, and then share your conjecture with a partner.
- Here's an exercise for you: If the diameter of the sphere above is 10 cm, what is the volume of the cylinder that encloses the sphere?
  - *Since the diameter of the sphere is* 10 cm*, the height of the cylinder is* 10 cm*, and the radius of the cylinder is* 5 cm*.*
  - *The base of the cylinder is a circle, so its area is* $\pi \cdot 5^2$*, or* $25\pi\text{ cm}^2$*.*
  - *Thus, the volume of the cylinder is* $25\pi \cdot 10$*, or* $250\pi\text{ cm}^3$*.*
- Finding the volume of the cylinder was straightforward. But finding the volume of the sphere is going to require some work!
- The ancient Greek mathematician Archimedes discovered the relationship between the volume of a sphere and the volume of a cylinder and was so proud of this achievement that he had the above figure etched into his tombstone. The key to his approach is to think of a sphere as a solid formed by many disks, as shown in the figure below. If we can somehow relate the size of each disk to the corresponding disks in the cylinder, we will know how the volumes of the two solids are related.



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Discussion (4 minutes): Cavalieri's Principle

- There is a general principle that can help us with this task. See if you can gain an understanding of this principle by studying the figure below. What do you notice? Take a moment to reflect on this image.

"Cavalieri's Principle in Coins" by Chiswick Chap is licensed under CC BY-SA 3.0
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MP.3

- Do the two stacks of coins have the same volume? How do you know?
  - *It appears that the two stacks contain exactly the same objects, so it makes sense to say that the two stacks have the same volume. It does not matter whether the coins are arranged in a regular fashion, as in the first image, or an irregular fashion, as in the second image.*
- What about the stacks shown below? Can you tell whether these have the same volume? Why or why not?

"Stacks of Coins" by Austin Kirk is licensed under CC BY 2.0
http://creativecommons.org/licenses/by/2.0

  - *Some of the coins are larger than the others. There is no easy way to tell whether the stacks have the same volume.*

- If you knew that two stacks both contained 10 quarters, 9 nickels, 8 pennies, and 7 dimes, do you feel confident that the stacks would have the same volume? Knowing the coins are the same size would be helpful. In a more general setting, we would like to know that the *cross sections* of two solids have the same area.
- Now, let's state the principle suggested by this discussion: Suppose two solids are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross sections of equal area, then the two regions have equal volumes.
- Can you understand the role that *parallel planes* play in this principle? For example, in the image below, where are the parallel planes that bound the two solids?

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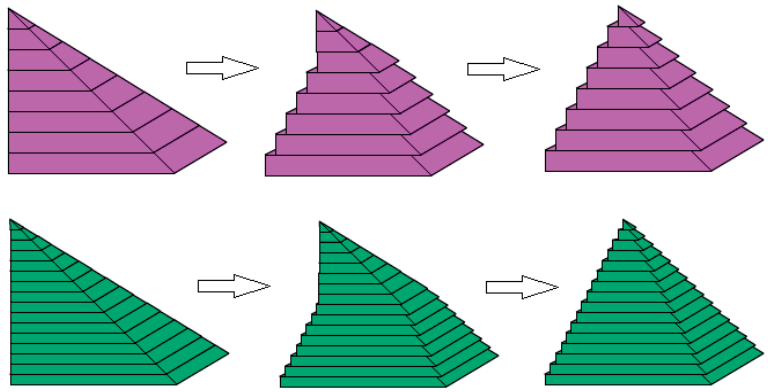
  - *The lower plane is the plane of the table on which the coins are resting. The upper plane is parallel to the plane of the tabletop and rests on the topmost coin in each stack.*
- Notice that when two solids are bounded by the same parallel planes, they are guaranteed to have the same height. Now, imagine the plane that lies halfway between these two boundary planes. Describe the intersection of this plane with the two stacks.
  - *Each plane that is parallel to the tabletop produces a cross section that is exactly equal in shape and size to the face of a coin. In particular, the area of a cross section is equal to the area of the face of a coin.*
- The idea we have been discussing is called Cavalieri's volume principle, which is named after an Italian mathematician who lived in the 17th century. But Archimedes was aware of this principle even in much more ancient times. We soon see how he used this volume principle to derive the relationship between the volume of a sphere and the volume of a cylinder.

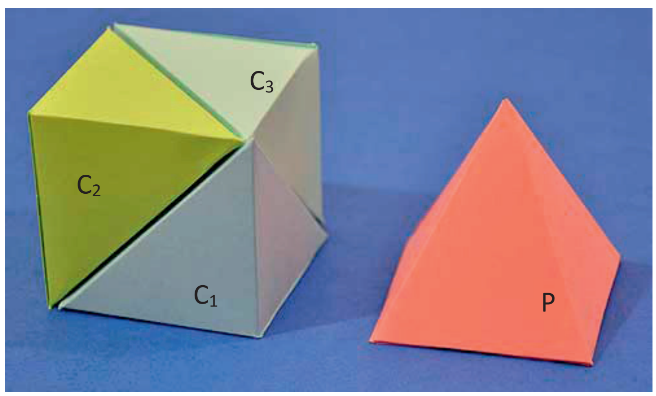
Discussion (4 minutes): The Volume of a Pyramid

- Here we can see how Cavalieri's principle applies to some pyramids. (This can be viewed as an animation at http://nrich.maths.org/7086&part=.) It can be shown that if two pyramids have the same base area and the same height, then they must have the same volume.



The NRICH website http://nrich.maths.org publishes free mathematics resources designed to challenge, engage, and develop the mathematical thinking of students aged 5 to 19.

- Note that the pyramids on the left can be arranged to form a cube:



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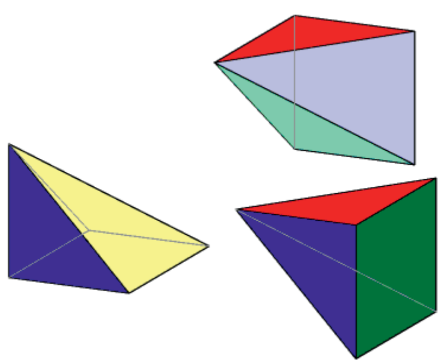
- Here is a challenge for you: If the edges of the cube on the left are 15 cm long, try to determine the volume of the pyramid on the right. Here is a decomposed picture of the cube to help you visualize what you are looking at



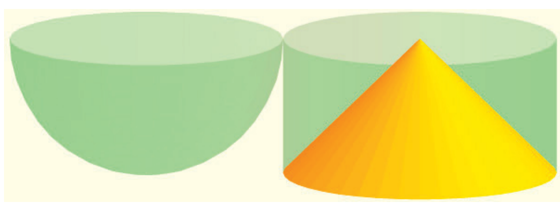
  - *If the edges of the cube are* 15 cm *long, then the volume of the cube is* $15\text{ cm} \times 15\text{ cm} \times 15\text{ cm}$*, or* 3375 cm^3.

- *The heights of each of the three pyramids must be equal since each of these is equal to the height of the cube. The base areas of each of the four pyramids must be equal since each of these is equal to the area of a face of the cube. Thus, all of the four pyramids have the same volume.*
- *It follows that each of the three pyramids on the left is one-third of the volume of the cube, which is* $\frac{3375 \text{ cm}^3}{3}$*, or* 1125 cm^3*. Thus, the volume of the pyramid above on the right of the cube is also* 1125 cm^3*.*

- The same reasoning can be used to show that any pyramid has one-third as much volume as a prism with the same base and the same height and that any cone has one-third as much volume as a cylinder with the same base and the same height.
- At this point, all of the groundwork has been laid. Let's see how Archimedes derived the formula for the volume of a sphere.

Discussion (6 minutes): Slicing a Hemisphere

- In the figure below, we see a hemisphere on the left and a cone on the right that is sitting inside a cylinder. The cylinder is just large enough to enclose the hemisphere. Our goal is to determine the relationships of the volumes of these three solids.



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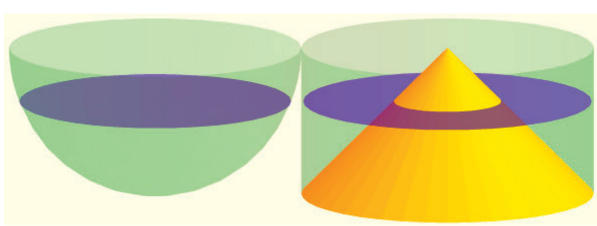
- We will suppose that the radius of the sphere is 5 cm. Next, we imagine that a plane is cutting through these solids, where the plane is parallel to the bases of the cylinder. Try to imagine what the cross sections look like. What do the cross sections of the sphere look like? What about the cylinder and the cone?
  - *All of the cross sections are circles.*



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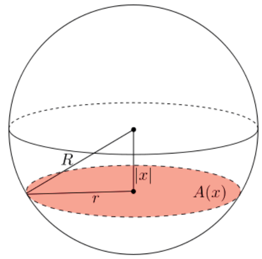
Scaffolding:

- If students are having trouble visualizing three-dimensional shapes, use cutouts from Geometry Module 3, or use common items such as balls, funnels, and disks that students can use to represent these shapes. Consider cutting some of these shapes apart so students can see the cross sections.
- The key to this task is to apply the Pythagorean theorem to a right triangle like the one shown in the drawing on the left. If students are struggling, provide them with a copy of this diagram, and ask, "What is the relationship between the lengths of the sides in a right triangle?"

- Let's see if we can compute the area of a few cross sections of these solids. Let x represent the distance between the slicing plane and the center of the sphere. If $x = 2 \text{ cm}$, what is the area of the blue disk on the left?

Give students several minutes to work on this task. Ask students to get into groups of three or four. Select a student or a group of students to present their work to the class at an appropriate time.



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- *The area of the circle is $A = \pi r^2$. To determine the area, we need to find the value of r.*
- *The plane cuts the sphere at $x = 2$ cm, and since the radius of the sphere is 5 cm, we have $(2\text{ cm})^2 + r^2 = (5\text{ cm})^2$. This means that $r^2 = 25\text{ cm}^2 - 4\text{ cm}^2 = 21\text{ cm}^2$, and so the area of the cross section is $21\pi\text{ cm}^2$.*

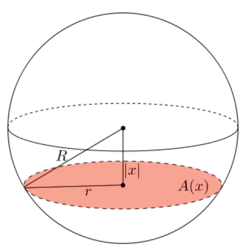
- Let's get some additional practice finding the areas of the cross sections of a sphere.

Exercise 1 (3 minutes)

Ask students to solve the following problems and to compare their results with a partner. Ask one or more students to present their solutions on the board.

Exercises

1. Let $R = 5$, and let $A(x)$ represent the area of a cross section for a circle at a distance x from the center of the sphere.



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a. Find $A(0)$. What is special about this particular cross section?

$A(0) = \pi \cdot 5^2 = 25\pi$. This is the largest cross section in the sphere; it's the area of a "great circle."

b. Find $A(1)$.

When $x = 1$, we have $r^2 + 1^2 = 5^2$. Thus, $A(1) = \pi \cdot r^2 = \pi \cdot (5^2 - 1^2) = \pi \cdot (25 - 1) = 24\pi$.

c. **Find $A(3)$.**

When $x = 3$, we have $r^2 + 3^2 = 5^2$. Thus, $A(3) = \pi \cdot r^2 = \pi \cdot (5^2 - 3^2) = \pi \cdot (25 - 9) = 16\pi$.

d. **Find $A(4)$.**

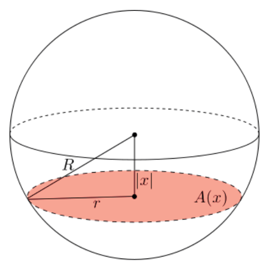
When $x = 4$, we have $r^2 + 4^2 = 5^2$. Thus, $A(4) = \pi \cdot r^2 = \pi \cdot (5^2 - 4^2) = \pi \cdot (25 - 16) = 9\pi$.

e. **Find $A(5)$. What is special about this particular cross section?**

$A(5) = 0$. When the plane reaches the point where $x = 5$, the cross section is a single point, so the area vanishes.

Discussion (6 minutes): Slicing the Cylinder and the Cone

- As we perform these calculations, a structure is beginning to emerge. Let's now turn our attention to the general case: Can you describe the area of a cross section formed by a plane cutting the sphere at a distance x from its center?



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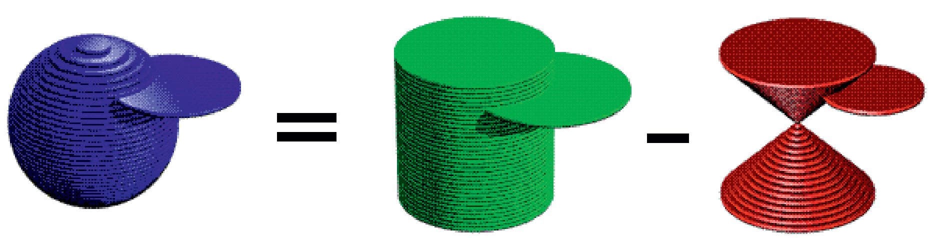
  - $A(x) = \pi \cdot r^2 = \pi \cdot ((5 \text{ cm})^2 - x^2) = \pi \cdot (25 \text{ cm}^2 - x^2)$

- Notice that we could use the distributive property to get $A(x) = \pi \cdot (25 \text{ cm}^2) - \pi \cdot x^2$. This looks like it could be the difference of two circles, and indeed it is. Momentarily, we show that the area of each disk in the sphere is equal to the difference in the areas of two other disks, as this diagram shows:



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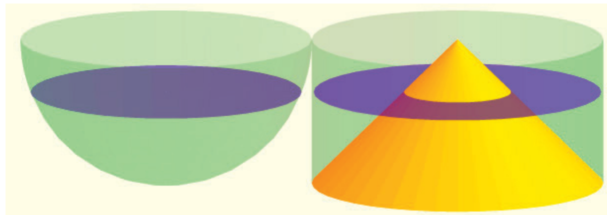
- Let's return now to the diagram that leads us to Archimedes' result:



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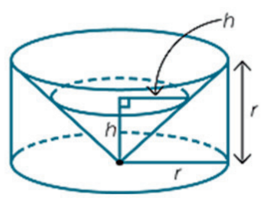
- We have analyzed the cross sections of the sphere; now, let's analyze the cross sections of the cone. In fact, let's focus on the blue ring surrounding the cone. If the slicing plane is h units below the top of the cylinder, what is the area of the blue ring in the figure on the right?

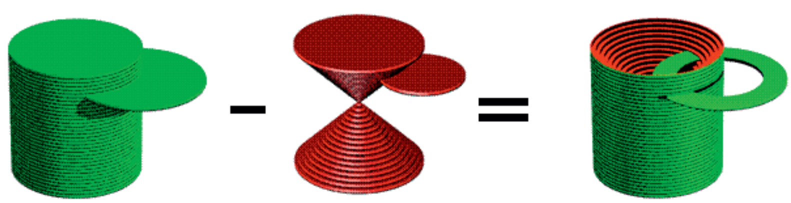
Give students several minutes to work on this task in groups. Select a student or a group of students to present their work to the class at an appropriate time.

Scaffolding:

- The key to this task is to recognize that the right triangle shown in the diagram to the left is similar to the triangle formed by the largest cone. Ask students, "How do we know when triangles are similar? What evidence do we have that the right triangles shown here are similar?"
- If students are struggling, provide them with a copy of this diagram.



  - *As the height of the slicing plane varies, the angles in the right triangle shown stay the same. Thus, a family of similar triangles is produced. Since the largest such triangle is an isosceles triangle with legs of length* 5*, it follows that all of the triangles are isosceles. So, at a distance* h *from the base of the cylinder, the radius of the cross section is* h *as well. Thus, the area of the cross section of the cone is* $\pi \cdot h^2$.
  - *The cross sections of the cylinder are uniform. Thus, for any height* h*, the area of a cross section of the cylinder is* $\pi \cdot (5\text{ cm})^2$*, or* $\pi \cdot 25\text{ cm}^2$.
- Our next task is to describe the space *around* the cone:



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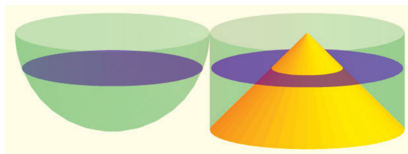
- If we take a section of this solid that is x units below the vertex of the double cone, then what is its area?
  - *The area of such a cross section is* $\pi \cdot 25\text{ cm}^2 - \pi \cdot x^2$.
- Does this look familiar? It is the same formula that describes the cross section of the sphere. Let's take a few minutes to confirm this result in specific cases.

Exercise 2 (2 minutes)

Ask students to solve the following problems and to compare their results with a partner. Ask one or more students to present their solutions on the board.

2. Let the radius of the cylinder be $R = 5$, and let $B(x)$ represent the area of the blue ring when the slicing plane is at a distance x from the top of the cylinder.



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a. **Find $B(1)$. Compare this area with $A(1)$, the area of the corresponding slice of the sphere.**

$B(1) = \pi \cdot 5^2 - \pi \cdot 1^2 = 25\pi - 1\pi = 24\pi$ *This is equivalent to $A(1)$.*

b. **Find $B(2)$. Compare this area with $A(2)$, the area of the corresponding slice of the sphere.**

$B(2) = \pi \cdot 5^2 - \pi \cdot 2^2 = 25\pi - 4\pi = 21\pi$ *This is equivalent to $A(2)$.*

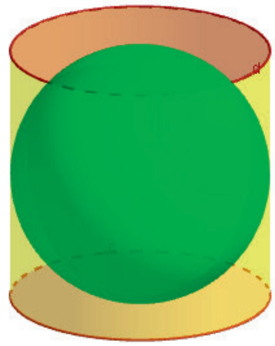
c. **Find $B(3)$. Compare this area with $A(3)$, the area of the corresponding slice of the sphere.**

$B(3) = \pi \cdot 5^2 - \pi \cdot 3^2 = 25\pi - 9\pi = 16\pi$ *This is equivalent to $A(3)$.*

Discussion (4 minutes): The Volume of a Sphere

- Now we have shown that the cross sections of the sphere are equal in area to the sections of the cylinder that lie outside the cone. What exactly does this prove about the solids themselves?
  - *Using Cavalieri's volume principle, we can conclude that the solids have equal volumes. That is, since their cross sections have equal areas and since the two solids are bounded between the same pair of parallel planes, they must have equal volumes.*

- How can we use the previous observation to compute the volume of the sphere with radius 5 cm? Take a minute to think about this.



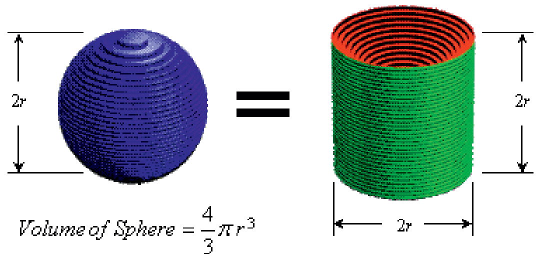
  - *We already found that the volume of the cylinder is* $250\pi\text{ cm}^3$.
  - *We know that a cone contains one-third as much volume as an enclosing cylinder, so the volume of the double cone inside the cylinder is* $\frac{1}{3}\cdot 250\pi\text{ cm}^3$.
  - *It follows that the space around the cone occupies the remaining two-thirds of the volume of the cylinder, which is* $\frac{2}{3}\cdot 250\pi\text{ cm}^3$.
  - *We proved that the volume of the hemisphere is equal to the volume of the space around the lower cone. It follows that the volume of the whole sphere is equal to the volume of the space around the double cone, which we just showed is* $\frac{2}{3}\cdot 250\pi\text{ cm}^3$. *This is the volume of the sphere with radius* 5 cm.
- Now would be a good time to revisit the balloon problem from the opening of the lesson. If a spherical balloon were pricked with a pin, allowing the water to leak out, how much of the cylinder would it fill? That's right. $66\frac{2}{3}\%$.
- The only thing left to do is to write a general formula for a sphere with radius r.

Exercise 3 (4 minutes)

Ask students to solve the following problem in their groups. Select a group to present its solution to the class.

MP.3

3. **Explain how to derive the formula for the volume of a sphere with radius r.**



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If we pass a plane through the sphere that is parallel to the bases of the cylinder at a distance x from the center of the sphere, we get a circle with area $A(x) = \pi \cdot (r^2 - x^2)$.

When the same plane intersects the cylinder, a ring is formed around the double cone. The area of this ring is $B(x) = \pi \cdot r^2 - \pi \cdot x^2$.

It's easy to see that $A(x) = B(x)$, and since both solids have height $2r$, it follows from Cavalieri's principle that the volume of the sphere is equal to the volume of the space outside the double cone.

The volume of the cylinder is $V = \pi \cdot r^2 \cdot 2r = 2\pi \cdot r^3$.

Thus, the volume of the double cone is $V = \frac{1}{3} \cdot 2\pi \cdot r^3$.

The volume of the space outside the double cone is therefore $V = \frac{2}{3} \cdot 2\pi \cdot r^3 = \frac{4}{3}\pi \cdot r^3$. Since the sphere also has this volume, this is the formula for the volume of the sphere.

- What do you make of this? James says that he prefers to think about the volume of a sphere using the formula $V = \frac{2}{3}Bh$. What do you suppose the variables B and h represent relative to the sphere? What do you suppose his rationale is for this preference?
  - *The cylinder that encloses the sphere has base area $B = \pi r^2$ and height $h = 2r$, and the gist of this lesson is that the sphere has two-thirds as much volume as the cylinder. The formula $V = \frac{2}{3}Bh$ makes all of these things visible.*

Closing (1 minutes)

- How are pyramids and prisms related with respect to their volumes? How are cones and cylinders related? How are spheres and cylinders related?
  - *A pyramid is one-third of a prism.*
  - *A cone is one-third of a cylinder.*
  - *A sphere is two-thirds of a cylinder.*

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 9: Volume and Cavalieri's Principle

Exit Ticket

Explain how Cavalieri's principle can be used to find the volume of any solid.

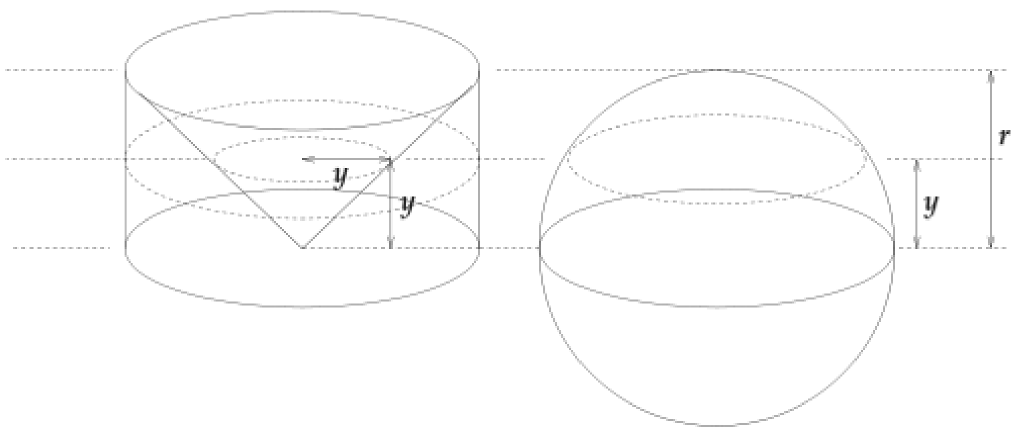
Exit Ticket Sample Solutions

Explain how Cavalieri's principle can be used to find the volume of any solid.

Cavalieri's principle tells us that to find the volume of a solid, we can examine cross sections of the solid. If another shape exists with the same height and equal areas of cross sections, then the two shapes will have equal volume.

Problem Set Sample Solutions

1. **Consider the sphere with radius $r = 4$. Suppose that a plane passes through the sphere at a height $y = 2$ units above the center of the sphere, as shown in the figure below.**



   a. **Find the area of the cross section of the sphere.**

   The sphere has radius 4, and the cross section passes through at $y = 2$, which tells us that the radius at the cross section is $r = \sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}$. Thus, the area of the cross section is 12π.

   b. **Find the area of the cross section of the cylinder that lies outside of the cone.**

   The cone has radius equal to its height at the cross section, and the circle passing through it has radius equal to 4 (since the radius of the cylinder is constant).

   c. **Find the volume of the cylinder, the cone, and the hemisphere shown in the figure.**

   The volume of the cylinder is $V = \pi r^2 h = 64\pi$ cubic units. The volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{64}{3}\pi$ cubic units. The volume of the hemisphere is twice the volume of the cone, so $\frac{128}{3}\pi$ cubic units.

   d. **Find the volume of the sphere shown in the figure.**

   The sphere is twice the volume of the hemisphere, so $\frac{256}{3}\pi$ cubic units.

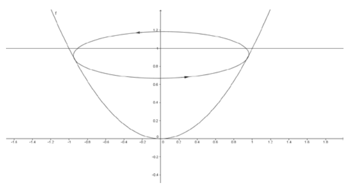
e. **Explain using Cavalieri's principle the formula for the volume of any single solid.**

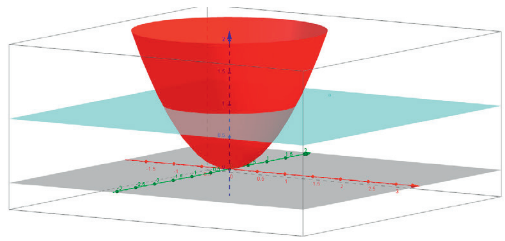
Cavalieri's principle tells us that to find the volume of a solid, we can examine cross sections of the solid. If another shape exists with the same height and equal areas of cross sections, then the two shapes have equal volume.

2. **Give an argument for why the volume of a right prism is the same as an oblique prism with the same height.**

Since the cross sections of both a right prism and an oblique prism would be the same shape (same area), it does not matter if the object is on a slant or straight up and down; they have the same volume.

3. **A paraboloid of revolution is a three-dimensional shape obtained by rotating a parabola around its axis. Consider the solid between a paraboloid described by the equation $y = x^2$ and the line $y = 1$.**





a. **Cross sections perpendicular to the y-axis of this paraboloid are what shape?**

Circles

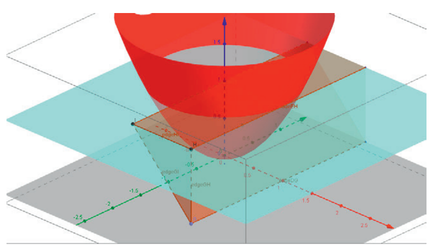
b. **Find the area of the largest cross section of this solid, when $y = 1$.**

When $y = 1, x = 1$, so $\pi 1^2 = \pi$.

c. **Find the area of the smallest cross section of this solid, when $y = 0$.**

When $y = 0$, $x = 0$, so $\pi 0^2 = 0$.

d. **Consider a right triangle prism with legs of length 1, hypotenuse of length $\sqrt{2}$, and depth π as pictured below. What shape are the cross sections of the prism perpendicular to the y-axis?**



Cross sections are rectangles.

e. **Find the areas of the cross sections of the prism at $y = 1$ and $y = 0$.**

At $y = 1$, the width is 1, and the depth is π, so $\pi \cdot 1 = \pi$.

At $y = 0$, the width is 0, and the depth is π, so $\pi \cdot 0 = 0$.

f. **Verify that at $y = y_0$, the areas of the cross sections of the paraboloid and the prism are equal.**

At y_0, the cross sections of the paraboloid have radius $x = \sqrt{y_0}$, so the area is $\pi\left(\sqrt{y_0}\right)^2 = y_0\pi$.

Similarly, the width of the rectangle is equal to the height of the prism, so at y_0, the width is y_0, and the depth is a constant π, so the area is $y_0\pi$.

g. **Find the volume of the paraboloid between $y = 0$ and $y = 1$.**

The volume of the paraboloid is equal to the volume of the right triangular prism, which has volume $V = \frac{1}{2}abh = \frac{1}{2} \cdot 1 \cdot 1 \cdot \pi = \frac{\pi}{2}$.

h. **Compare the volume of the paraboloid to the volume of the smallest cylinder containing it. What do you notice?**

The volume of the paraboloid is half the volume of the cylinder.

i. **Let V_{cyl} be the volume of a cylinder, V_{par} be the volume of the inscribed paraboloid, and V_{cone} be the volume of the inscribed cone. Arrange the three volumes in order from smallest to largest.**

$V_{\text{cone}} < V_{\text{par}} < V_{\text{cyl}}$ since $V_{\text{cone}} = \frac{1}{3}V_{\text{cyl}}$ and $V_{\text{par}} = \frac{1}{2}V_{\text{cyl}}$

4. **Consider the graph of f described by the equation $f(x) = \frac{1}{2}x^2$ for $0 \leq x \leq 10$.**

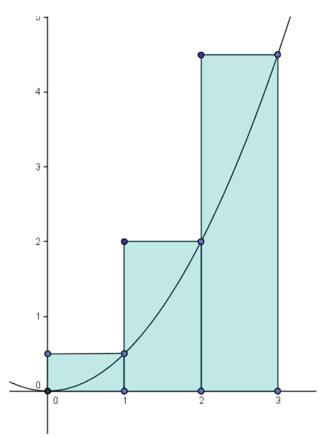
a. **Find the area of the 10 rectangles with height $f(i)$ and width 1, for $i = 1, 2, 3, \dots, 10$.**

In each case, the width is one, so the area is equal to the height of the function at that point; we get $\frac{1}{2}, 2, \frac{9}{2}, 8, \frac{25}{2}, 18, \frac{49}{2}, 32, \frac{81}{2}, 50$.

b. **What is the total area for $0 \leq x \leq 10$? That is, evaluate $\sum_{i=1}^{10} f(i) \cdot \Delta x$ for $\Delta x = 1$.**

$$\sum_{i=1}^{10} f(i) \cdot \Delta x = 192.5$$

c. **Draw a picture of the function and rectangles for $i = 1, 2, 3$.**



d. **Is your approximation an overestimate or an underestimate?**

Since each rectangle contains more area than is under the parabola, the estimate is an overestimate.

e. **How could you get a better approximation of the area under the curve?**

Answers may vary and may include that you could find smaller rectangles, you could find the underestimate by using the left endpoints, and you could cut off the triangles above the function to get a trapezoid.

5. **Consider the three-dimensional solid that has square cross sections and whose height y at position x is given by the equation $y = 2\sqrt{x}$ for $0 \leq x \leq 4$.**

a. **Approximate the shape with four rectangular prisms of equal width. What is the height and volume of each rectangular prism? What is the total volume?**

The heights are 2, $2\sqrt{2}$, $2\sqrt{3}$, 4, and the volumes are 4, 8, 12, 16. The total volume is 40.

b. **Approximate the shape with eight rectangular prisms of equal width. What is the height and volume of each rectangular prism? What is the total volume?**

The heights are $2\sqrt{\frac{1}{2}}$, 2, $2\sqrt{\frac{3}{2}}$, $2\sqrt{2}$, $2\sqrt{\frac{5}{2}}$, $2\sqrt{3}$, $2\sqrt{\frac{7}{2}}$, 4, and the volumes are 1, 2, 3, 4, 5, 6, 7, 8. The total volume is 36.

c. **How much did your approximation improve? The volume of the shape is 32 cubic units. How close is your approximation from part (b)?**

The approximation improved by 4 cubic units. The approximation is off by 4 cubic units.

d. **How many rectangular prisms would you need to be able to approximate the volume accurately?**

It is hard to say, but many would be needed to get significant accuracy, although this could be reduced by taking both an upper and a lower bound.

Name ____ Date ____________________

1.

a. Write $(1+i)^7-(1-i)^7$ in the form $a+bi$ for some real numbers a and b.

b. Explain how Pascal's triangle allows you to compute the coefficient of x^2y^3 when $(x-y)^5$ is expanded.

2. Verify that the fundamental theorem of algebra holds for the fourth-degree polynomial p given by $p(z) = z^4 + 1$ by finding four zeros of the polynomial and writing the polynomial as a product of four linear terms. Be sure to make use of the polynomial identity given below.

$$x^4 - a^4 = (x-a)(x+a)(x-ai)(x+ai)$$

3. Consider the cubic polynomial p given $p(z) = z^3 - 8$.

   a. Find a real number root to the polynomial.

   b. Write $p(z)$ as a product of three linear terms.

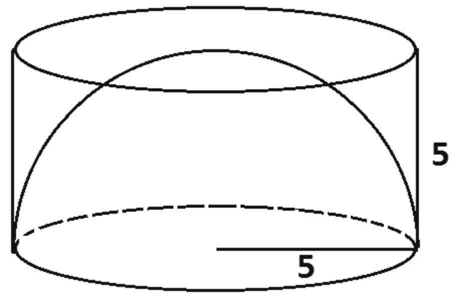
Consider the degree-eight polynomial q given by $q(z) = z^8 - 2^8$.

c. What is the largest possible number of distinct roots the polynomial q could possess? Briefly explain your answer.

d. Find all the solutions to $q(z) = 0$.

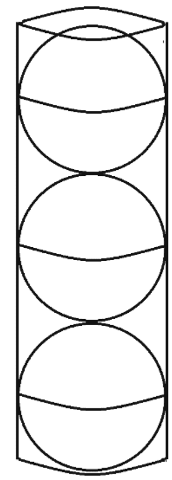
4.

a. A right circular cylinder of radius 5 cm and height 5 cm contains half a sphere of radius 5 cm as shown.



Use Cavalieri's principle to explain why the volume inside this cylinder but outside the hemisphere is equivalent to the volume of a circular cone with base of radius 5 cm and height 5 cm.

b. Three congruent solid balls are packaged in a cardboard cylindrical tube. The cylindrical space inside the tube has dimensions such that the three balls fit snugly inside that tube as shown.



Each ball is composed of material with density 15 grams per cubic centimeter. The space around the balls inside the cylinder is filled with aerated foam with a density of 0.1 grams per cubic centimeter.

i. Ignoring the cardboard of the tube, what is the average density of the contents inside of the tube?

ii. If the contents inside the tube, the three balls and the foam, weigh 150 grams to one decimal place, what is the weight of one ball in grams?

5.

a. Consider the two points $F(-9, 0)$ and $G(9, 0)$ in the coordinate plane. What is the equation of the ellipse given as the set of all points P in the coordinate plane satisfying $FP + PG = 30$? Write the equation in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with a and b real numbers, and explain how you obtain your answer.

b. Consider again the two points $F(-9, 0)$ and $G(9, 0)$ in the coordinate plane. The equation of the hyperbola defined by $|FP - PG| = k$ for some constant k is given by $\frac{x^2}{25} - \frac{y^2}{56} = 1$. What is the value of k?

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	**STEP 2** Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	**STEP 3** A correct answer with some evidence of reasoning or application of mathematics to solve the problem, OR an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	**STEP 4** A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	**a** A-APR.C.5	Student shows little or no understanding of the binomial theorem or Pascal's triangle.	Student expands one binomial correctly.	Student expands both binomials correctly but makes a mistake in calculating the final answer.	Student expands both binomials correctly and determines the correct final answer in $a + bi$ form.
	b A-APR.C.5	Student shows little or no understanding of the binomial theorem or Pascal's triangle.	Student knows that the fifth row of Pascal's triangle is needed to complete the expansion.	Student shows the correct expansion but makes a sign error.	Student shows the correct expansion and identifies the correct coefficient.
2	N-CN.C.8 N-CN.C.9	Student shows little or no understanding of the fundamental theorem of algebra.	Student begins factoring the polynomial but makes mistakes.	Student finds the four zeros of the polynomial but does not show the final factored form. OR Student shows the final factored form of the polynomial but does not find the zeros.	Student finds the four zeros of the polynomial and writes the final factored form correctly.
3	**a** N-CN.C.8 N-CN.C.9	Student shows little or no knowledge of roots of polynomials or factoring.	Student identifies a real root, but it is not correct.	Student identifies more than one real root with one being correct.	Student correctly identifies one real root.

	b **N-CN.C.8** **N-CN.C.9**	Student shows little or no knowledge of factoring polynomials.	Student factors into the correct binomial and trinomial.	Student factors into the correct binomial and trinomial but makes a minor mathematical mistake when factoring the trinomial further.	Student factors the polynomial correctly.
	c **N-CN.C.8** **N-CN.C.9**	Student shows little or no knowledge of the fundamental theorem of algebra.	Student shows some knowledge of the number of roots but does not state the correct number of roots.	Student states the correct number of roots but does not explain why based on the fundamental theorem of algebra.	Student states the correct number of roots and explains that this is a condition of the fundamental theorem of algebra.
	d **N-CN.C.8** **N-CN.C.9**	Student shows little or no knowledge of roots or factoring polynomials.	Student factors correctly but identifies fewer than four correct roots.	Student factors correctly but only identifies four correct roots.	Student factors correctly and identifies eight correct roots.
4	**a** **G-GMD.A.2**	Student shows little or no knowledge of Cavalieri's principle.	Student attempts to explain using Cavalieri's principle but makes major mathematical mistakes.	Student uses Cavalieri's principle correctly but does not answer the question completely.	Student uses Cavalieri's principle correctly and fully answers the question.
	b **G-GMD.A.2**	Student shows little or no knowledge of Cavalieri's principle.	Student attempts to explain using the results of part (a) and Cavalieri's principle but makes major mathematical mistakes.	Student uses the results of part (a) and Cavalieri's principle correctly but makes a small error in calculating the ratio.	Student uses the results of part (a) and Cavalieri's principle to correctly explain and calculate the ratio.
5	**a** **G-GPE.A.3**	Student shows little or no knowledge of ellipses.	Student shows some knowledge of ellipses and determines the correct value of k but does not write the equation of the ellipse.	Student determines k correctly and writes the equation of the ellipse but reverses a and b.	Student determines k correctly and writes the correct equation of the ellipse.
	b **G-GPE.A.3**	Student shows little or no knowledge of hyperbolas.	Student shows some knowledge of hyperbolas but makes mistakes in determining points needed to write the equation.	Student writes the equation of the hyperbola but switches x and y or reverses a and b.	Student writes the correct equation of the hyperbola.

Name ______________________________ Date ______________

1.

a. Write $(1+i)^7 - (1-i)^7$ in the form $a + bi$ for some real numbers a and b.

The seventh row of Pascal's triangle is 1 7 21 35 35 21 7 1. Thus:

$$\begin{aligned}(1+i)^7 &= 1 + 7i + 21i^2 + 35i^3 + 35i^4 + 21i^5 + 7i^6 + i^7 \\ &= 1 + 7i - 21 - 35i + 35 + 21i - 7 - i\end{aligned}$$

and

$$\begin{aligned}(1-i)^7 &= 1 - 7i + 21i^2 - 35i^3 + 35i^4 - 21i^5 + 7i^6 - i^7 \\ &= 1 - 7i - 21 + 35i + 35 - 21i - 7 + i.\end{aligned}$$

Their difference is

$$(1+i)^7 - (1-i)^7 = 14i - 70i + 42i - 2i = -16i.$$

This answer is in the form $a + bi$ with $a = 0$ and $b = -16$.

b. Explain how Pascal's triangle allows you to compute the coefficient of x^2y^3 when $(x-y)^5$ is expanded.

The fifth row of Pascal's triangle is 1 5 10 10 5 1. Thus:

$$\begin{aligned}(x-y)^5 &= \left(x + (-y)\right)^5 \\ &= x^5 + 5x^4(-y) + 10x^3(-y)^2 + 10x^2(-y)^3 + 5x(-y)^4 + (-y)^5 \\ &= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5\end{aligned}$$

The coefficient of x^2y^3 is -10.

2. Verify that the fundamental theorem of algebra holds for the fourth-degree polynomial p given by $p(z) = z^4 + 1$ by finding four zeros of the polynomial and writing the polynomial as a product of four linear terms. Be sure to make use of the polynomial identity given below.

$$x^4 - a^4 = (x-a)(x+a)(x-ai)(x+ai)$$

We have $p(z) = z^4 - (-1)$ suggesting we need to find a number a so that $a^4 = -1$. This means $a^2 = i$ or $a^2 = -i$. Since we need to find only one value for a that works, let's select $a^2 = i$.

Now i has modulus 1 and argument $\frac{\pi}{2}$, so a complex number a with modulus 1 and argument $\frac{\pi}{4}$ satisfies $a^2 = i$. So, $a = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1+i}{\sqrt{2}}$.

(And we check: $\left(\frac{1+i}{\sqrt{2}}\right)^2 = \frac{1+2i-1}{2} = i$.)

So, $p(z) = z^4 - \left(\frac{1+i}{\sqrt{2}}\right)^4 = \left(z - \frac{1+i}{\sqrt{2}}\right)\left(z + \frac{1+i}{\sqrt{2}}\right)\left(z - \frac{1-i}{\sqrt{2}}\right)\left(z + \frac{1-i}{\sqrt{2}}\right)$ following the polynomial identity given. Thus, we see that p does indeed factor into four linear terms and has four roots: $\frac{1+i}{\sqrt{2}}$, $-\frac{1+i}{\sqrt{2}}$, $\frac{1-i}{\sqrt{2}}$, and $-\frac{1-i}{\sqrt{2}}$.

3. Consider the cubic polynomial p given $p(z) = z^3 - 8$.

   a. Find a real number root to the polynomial.

   $z = 2$ is a root.

   b. Write $p(z)$ as a product of three linear terms.

   We have $z^3 - 8 = (z-2)(z^2 + 2z + 4)$.

   Now $z^2 + 2z + 4 = 0$ when $z = \frac{-2 \pm \sqrt{4 - 16}}{2} = -1 \pm \sqrt{3}i$ showing that $z^2 + 2z + 4$ factors as $(z + 1 + \sqrt{3}i)(z + 1 - \sqrt{3}i)$.

   Thus:

   $$p(z) = (z-2)(z+1+\sqrt{3}i)(z+1-\sqrt{3}i).$$

Consider the degree-eight polynomial q given by $q(z) = z^8 - 2^8$.

c. What is the largest possible number of distinct roots the polynomial q could possess? Briefly explain your answer.

By the fundamental theorem of algebra, a degree-eight polynomial has at most 8 distinct roots.

d. Find all the solutions to $q(z) = 0$.

We have

$$\begin{aligned} q(z) &= z^8 - 2^8 \\ &= (z^4 - 2^4)(z^4 + 2^4) \\ &= (z^2 - 2^2)(z^2 + 2^2)(z^4 + 2^4) \\ &= (z-2)(z+2)(z-2i)(z+2i)(z^2 - 4i)(z^2 + 4i). \end{aligned}$$

We see the zeros:

$z = 2$, $z = -2$, $z = 2i$, *and* $z = -2i$.

Going further, we need to also solve $z^2 - 4i = 0$ *and* $z^2 + 4i = 0$.

Now if $(a+bi)^2 = 4i$, *we have* $a^2 - b^2 = 0$ *(giving* $a = \pm b$*) and* $2ab = 4$ *(i.e.,* $ab = 2$*). If* $a = b$, *we get* $a^2 = 2$; *so,* $a = b = \sqrt{2}$ *or* $a = b = -\sqrt{2}$. *If* $a = -b$ *we get* $a^2 = -2$, *which has no solution. So we see*

If $z^2 - 4i = 0$, *then* $z = \sqrt{2} + \sqrt{2}i$ *or* $z = -\sqrt{2} - \sqrt{2}i$.

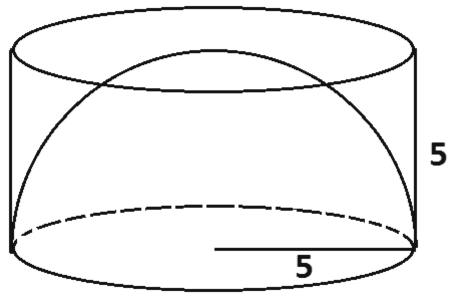
The same work shows the following:

If $z^2 + 4i = 0$, *then* $z = \sqrt{2} - \sqrt{2}i$ *or* $z = -\sqrt{2} + \sqrt{2}i$.

We have thus identified the eight zeros of q.

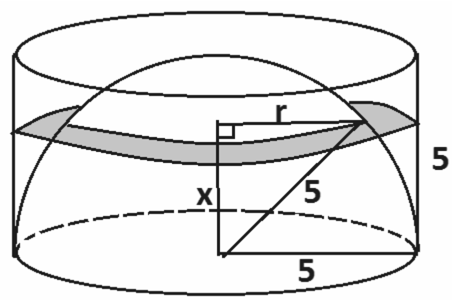
4.

a. A right circular cylinder of radius 5 cm and height 5 cm contains half a sphere of radius 5 cm as shown.



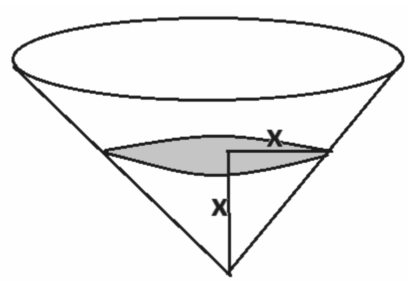
Use Cavalieri's principle to explain why the volume inside this cylinder but outside the hemisphere is equivalent to the volume of a circular cone with base of radius 5 cm and height 5 cm.

Look at a horizontal cross-section of the region inside the cylinder but outside the hemisphere. It is ring-shaped—the region between two circular discs.



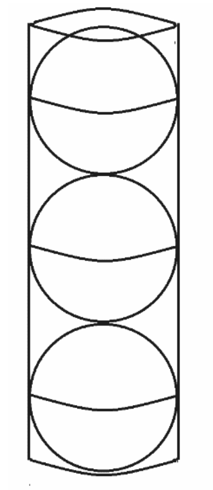
If the height of the cross-section in centimeters is x as shown ($0 \le x \le 5$), and the length r in centimeters is the distance from the vertical line of symmetry of the figure to the surface of the hemisphere as shown, then the area of the horizontal cross-section in square centimeters is $\pi 5^2 - \pi r^2$, or $\pi(25 - r^2)$. By the Pythagorean theorem, (look at another radius of the sphere) this equals πx^2, which is the area of a circle of radius x.

If we draw the solid figure whose horizontal cross-section at height x in centimeters for $0 \le x \le 5$ is a circle of radius x, we get a circular cone of height 5 cm and base radius of 5 cm.



By Cavalieri's principle, the volume of the region inside the cylinder but outside the hemisphere is equivalent to the volume of this circular cone.

b. Three congruent solid balls are packaged in a cardboard cylindrical tube. The cylindrical space inside the tube has dimensions such that the three balls fit snugly inside that tube as shown.



Each ball is composed of material with density 15 grams per cubic centimeter. The space around the balls inside the cylinder is filled with aerated foam with a density of 0.1 grams per cubic centimeter.

i. Ignoring the cardboard of the tube, what is the average density of the inside contents of the tube?

ii. If the contents inside the tube, the three balls and the foam, weigh 150 grams to one decimal place, what is the weight of one ball in grams?

From part (a), since the volume of a cone is one-third the volume of a cylinder with the same base and same height, the space inside the cylinder and outside the hemisphere in that question is one-third the volume of the cylinder. This means that the volume of the hemisphere is double the volume of this space.

For three balls packed in a cylinder, we have six copies of the situation analyzed in part (a). Thus, the volume of foam inside the package and outside of the balls is one-half the total volume of the balls.

Let V_f denote the total volume of the foam and V_b the total volume of the balls. Then we have $V_f = \frac{1}{2}V_b$, or $V_b = 2V_f$.

Let M_f be the total mass of the foam and M_b be the total mass of the balls. Density is mass per volume, so the densities in grams per cubic centimeter are as follows:

density of the foam: $\frac{M_f}{V_f} = 0.1$

density of the balls: $\frac{M_b}{V_b} = 15.$

i)

$$\begin{aligned}
\text{Density}_{average} &= \frac{M_f + M_b}{V_f + V_b} \\
&= \frac{M_f + M_b}{3V_f} \\
&= \frac{1}{3} \cdot \frac{M_f}{V_f} + \frac{M_b}{3V_f} \\
&= \frac{1}{3}(0.1) + \frac{2}{3} \cdot \frac{M_b}{V_b} \\
&= \frac{1}{3}(0.1) + \frac{2}{3}(15) \\
&= \frac{1}{30} + 10 \\
&\approx 10.033
\end{aligned}$$

The average density of the contents is approximately 10.033 grams/cubic centimeter.

ii) As the weight of the foam is negligible, we expect each ball to weigh approximately 50 grams. To get the exact weight, use

Total mass = total volume × average density.

$$\left(10 + \frac{1}{30}\right) \times (V_f + V_b) = 150$$

$$\left(10 + \frac{1}{30}\right) \times \frac{3}{2}V_b = 150$$

$$V_b = \frac{100}{10 + \frac{1}{30}} = \frac{3000}{301}$$

Since we have 3 balls, we must divide by 3: $\frac{1}{3} \times \frac{3000}{301} = \frac{1000}{301}$. The volume of one ball is $\frac{1000}{301}$ cubic centimeters.

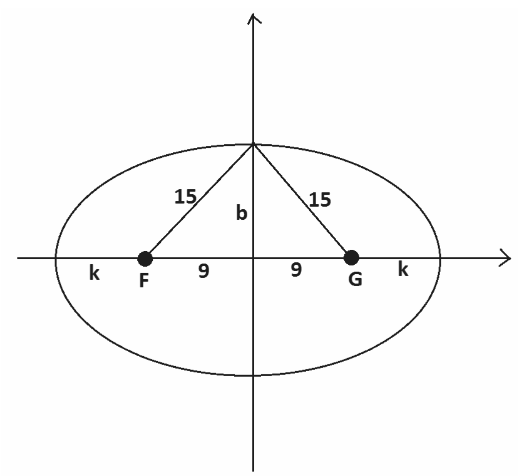
As the density of each ball is 15 grams per cubic centimeter: $15 \times \frac{1000}{301} \approx 49.8$.

The weight of one ball is approximately 49.8 grams.

5.

a. Consider the two points $F(-9, 0)$ and $G(9, 0)$ in the coordinate plane. What is the equation of the ellipse given as the set of all points P in the coordinate plane satisfying $FP + PG = 30$? Write the equation in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with a and b real numbers, and explain how you obtain your answer.

Suppose the ellipse described crosses the positive x-axis at $x = a$ and the positive y-axis at b. Let k be the distance between G and the positive x-intercept as shown.



For the point $P(0, b)$ on the ellipse, we have $FP + PG = 30$. By symmetry, this means $PG = 15$, and by the Pythagorean theorem, $b = \sqrt{15^2 - 9^2} = 12$.

For the point $Q(a, 0)$ on the ellipse, we have $GP + PF = 18 + 2k = 30$, giving $k = 6$ and $a = 9 + k = 15$. Thus, the equation of the ellipse is $\frac{x^2}{15^2} + \frac{y^2}{12^2} = 1$.

OR

For any point $P(x, y)$ on the ellipse:

$$FP = \sqrt{(x+9)^2 + y^2}$$

$$PG = \sqrt{(x-9)^2 + y^2}$$

and $FP + PG = 30$ reads

$$\sqrt{(x+9)^2 + y^2} + \sqrt{(x-9)^2 + y^2} = 30.$$

This can be rewritten as follows:

$$\sqrt{(x-9)^2 + y^2} = 30 - \sqrt{(x+9)^2 + y^2}.$$

Squaring gives the following:

$$(x-9)^2 + y^2 = 900 + (x+9)^2 + y^2 - 60\sqrt{(x+9)^2 + y^2}$$

$$-18x = 900 + 18x - 60\sqrt{(x+9)^2 + y^2}$$

$$60\sqrt{(x+9)^2 + y^2} = 900 + 36x$$

$$10\sqrt{(x+9)^2 + y^2} = 150 + 6x.$$

Squaring one more time produces the following:

$$100\left((x+9)^2 + y^2\right) = 22500 + 36x^2 + 1800x$$

$$100x^2 + 1800x + 8100 + 100y^2 = 22500 + 36x^2 + 1800x$$

$$64x^2 + 100y^2 = 14400$$

$$\frac{x^2}{225} + \frac{y^2}{144} = 1.$$

Thus, any point $P(x, y)$ on the ellipse must be a solution to the equation

$$\frac{x^2}{15^2} + \frac{y^2}{12^2} = 1.$$

b. Consider again the two points $F(-9, 0)$ and $G(9, 0)$ in the coordinate plane. The equation of the hyperbola defined by $|FP - PG| = k$ for some constant k is given by $\frac{x^2}{25} - \frac{y^2}{56} = 1$. What is the value of k?

Consider the equation

$$\frac{x^2}{25} - \frac{y^2}{56} = 1.$$

Setting $y = 0$ shows that $P(5, 0)$ is a point on the hyperbola. Then $FP = 4$ and $PG = 14$, so $k = |FP - PG| = 10$.

Topic B

Rational Functions and Composition of Functions

A-APR.D.7, F-IF.C.7d, F-IF.C.9, F-BF.A.1c

Focus Standards:	A-APR.D.7	(+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.
	F-IF.C.7	Graph functions expressed symbolically and show key features of the graph by hand in simple cases and using technology for more complicated cases. d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.
	F-IF.C.9	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*
	F-BF.A.1	Write a function that describes a relationship between two quantities. c. (+) Compose functions. *For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.*
Instructional Days:	8	
Lesson 10:	The Structure of Rational Expressions (P)[1]	
Lesson 11:	Rational Functions (P)	
Lesson 12:	End Behavior of Rational Functions (P)	
Lesson 13:	Horizontal and Vertical Asymptotes of Graphs of Rational Functions (P)	
Lesson 14:	Graphing Rational Functions (P)	
Lesson 15:	Transforming Rational Functions (E)	
Lesson 16:	Function Composition (P)	
Lesson 17:	Solving Problems by Function Composition (P)	

[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Previous courses have introduced students to working with rational expressions. Lesson 10 focuses on developing the idea that rational expressions form a system analogous to the rational numbers. Students use the properties of rational numbers to explore rational expressions and see that operations on rational expressions are defined in the same way as operations on rational numbers. In particular, students use the properties of integers to establish closure for the set of rational numbers. Students then use properties of polynomials to establish closure for the set of rational expressions (**A-APR.D.7**).

In Module 1 of Algebra II, students simplified rational expressions and performed arithmetic operations with them, which prepared them to solve rational equations. Lesson 11 revisits this process, helping students to form conjectures about the closure property for rational functions under arithmetic operations (**A-APR.D.7**), which leads to function composition. They review simplifying rational expressions with a focus on restricted domain values and then compare the properties of rational functions represented in different ways (**F-IF.C.9**). In Lesson 12, students look at the end behavior of rational functions numerically.

Lesson 13 defines horizontal and vertical asymptotes. While students saw vertical asymptotes in Algebra II when graphing the tangent function, this is the first time that they encounter the formal definition of vertical asymptotes. Students determine horizontal and vertical asymptotes of rational functions and use technology to confirm their findings. In Lesson 14, students analyze the key features of a rational function including zeros, intercepts, asymptotes, and end behavior, and then they graph rational functions without the aid of technology (**F-IF.C.7d**). Lesson 15 extends students' work on graphs of rational functions to include transformations (**F-IF-C.7d**).

Lesson 16 explores functions and their compositions, including situations where the sets representing the inputs and outputs may not be numerical. Students find the composition of functions in real-world contexts and assess the reasonableness of the compositions (**F-BF.A.1c**). Topic B concludes with Lesson 17 as students focus on composing numerical functions, including those in real-world context. Students represent real-world relationships with equations, use those equations to create composite functions, and then use the composite functions to solve problems in both mathematical and real-world contexts (**F-IF.C.9**). Through this work, students see that some compositions do not make sense or are not possible, depending on the context.

In Topic B, students use technology as a tool to understand key features of graphs (MP.5). They relate the structure of rational expressions to the graphs of rational functions in studying transformations of these graphs (MP.7). Students use mathematics to model (MP.4) as they write formulas for the surface area of spheres in terms of their diameters. They then use functions and function composition to study the relationship between a deep-sea diver's depth, atmospheric pressure, and time.

Lesson 10: The Structure of Rational Expressions

Student Outcomes

- Students understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression.
- Students add, subtract, multiply, and divide rational expressions.

Lesson Notes

In Algebra II, students performed operations on rational expressions. They learned that the process of combining rational expressions is analogous to that of combining rational numbers. While this lesson lets students review these skills, the focus here is on understanding that rational expressions form a system analogous to the rational numbers. In particular, students use the properties of integers to establish closure for the set of rational numbers, and then they use properties of polynomials to establish closure for the set of rational expressions.

Note: Students are often directed to simplify rational expressions, which may require them to add, subtract, multiply, or divide two rational expressions and to reduce the resulting expression by dividing out common factors. The term *simplify* can prove problematic because it is not always clear whether the rational expression that results from the procedures above is simpler than the original expression. The goal is for students to write the rational expression so that there is a single polynomial denominator.

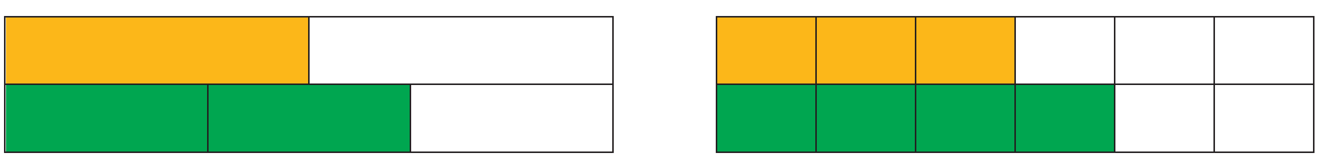
Classwork

Opening Exercise (6 minutes)

This lesson reviews what students learned in Algebra II about how to add, subtract, multiply, and divide rational expressions. Then, a connection is established between the properties of rational expressions and those of rational numbers. In this set of exercises, students perform addition and subtraction: first with rational numbers and then with rational expressions.

The bar model below for $\frac{1}{2}+\frac{2}{3}$ can be presented to students as scaffolding if they need a reminder on how to add fractions.

The following representation shows that $\frac{1}{2}+\frac{2}{3}=\frac{1}{2}\cdot\frac{3}{3}+\frac{2}{3}\cdot\frac{2}{2}=\frac{3}{6}+\frac{4}{6}=\frac{7}{6}$.



Opening Exercise

a. **Add the fractions:** $\frac{3}{5}+\frac{2}{7}$.

$$\frac{3}{5}+\frac{2}{7}=\frac{3}{5}\cdot\frac{7}{7}+\frac{2}{7}\cdot\frac{5}{5}=\frac{21}{35}+\frac{10}{35}=\frac{31}{35}$$

b. **Subtract the fractions:** $\frac{5}{2}-\frac{4}{3}$.

$$\frac{5}{2}-\frac{4}{3}=\frac{5}{2}\cdot\frac{3}{3}-\frac{4}{3}\cdot\frac{2}{2}=\frac{15}{6}-\frac{8}{6}=\frac{7}{6}$$

c. **Add the expressions:** $\frac{3}{x}+\frac{x}{5}$.

$$\frac{3}{x}+\frac{x}{5}=\frac{3}{x}\cdot\frac{5}{5}+\frac{x}{5}\cdot\frac{x}{x}=\frac{15}{5x}+\frac{x^2}{5x}=\frac{15+x^2}{5x}$$

d. **Subtract the expressions:** $\frac{x}{x+2}-\frac{3}{x+1}$.

$$\frac{x}{x+2}-\frac{3}{x+1}=\frac{x}{x+2}\cdot\frac{x+1}{x+1}-\frac{3}{x+1}\cdot\frac{x+2}{x+2}=\frac{x^2+x}{(x+2)(x+1)}-\frac{3x+6}{(x+1)(x+2)}=\frac{(x^2+x)-(3x+6)}{(x+2)(x+1)}$$
$$=\frac{x^2-2x-6}{(x+2)(x+1)}$$

Scaffolding:

Give cues to students as necessary using questions such as these:

- Do the fractions have a common denominator?
- What operations could be performed to get a common denominator?
- Demonstrate fraction addition using a bar model if necessary.

Discussion (5 minutes)

This Discussion should lead students to consider the idea of closure and the connections between the structure of operations performed with rational numbers to those of operations performed with rational expressions.

MP.7

- How are Exercises 3 and 4 similar to Exercises 1 and 2? How are they different?
  - *In both cases, we need a common denominator in order to combine the expressions to form a single entity.*
  - *In the case of Exercises 1 and 2, the results are numbers, but in Exercises 3 and 4, the results are expressions that contain a variable.*
- In Exercise 1, you found that $\frac{3}{5}+\frac{2}{7}=\frac{31}{35}$. In Exercise 2, you found that $\frac{5}{2}-\frac{4}{3}=\frac{7}{6}$. What do these exercises illustrate about the sum and difference of rational numbers?
  - *The sum or difference of two rational numbers is a rational number.*
- Let's review why this is true. How do we define a rational number?
  - *It is the ratio of two integers, where the denominator does not equal zero.*

MP.3

- How can we reason why the sum or difference of two rational numbers is rational?
  - *Finding the numerator of the sum or difference of rational numbers requires us to add, subtract, and/or multiply integers. We find the denominator by multiplying integers. The product of two integers is an integer; likewise, the sum and difference of two integers is an integer. Therefore, the numerators and denominators are both integers, which means that the sum or difference is a rational number.*
  - *The denominator is the product of two nonzero integers, so the product cannot be zero.*

- What is the mathematical word for this property, and where have we used it before?
  - *Closure: Students may recall discussing closure with respect to integers, rational numbers, and polynomials.*

Exercises 1–2 (8 minutes)

In these exercises, students use the technique shown above to construct an argument. Provide cues as needed to help them develop this argument. Encourage students to work together. Let them work for several minutes, and then select a student to present his argument to the class.

Exercises

1. **Construct an argument that shows that the set of rational numbers is closed under addition. That is, if x and y are rational numbers and $w = x + y$, prove that w must also be a rational number.**

   Since x and y are rational numbers, there are four integers, a, b, c, and d, with $x = \frac{a}{b}$ and $y = \frac{c}{d}$, and neither b nor d is 0.

   Now we need to check to see if w is a rational number:
   $$w = x + y = \frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b} = \frac{ad + cb}{bd}$$
   The numerator is formed by multiplying and adding integers, so it must be an integer. Similarly, the denominator must be an integer. Lastly, bd cannot be 0 since neither b nor d is 0. This proves that w is a rational number.

2. **How could you modify your argument to show that the set of rational numbers is also closed under subtraction? Discuss your response with another student.**

   This time, we start with $\frac{a}{b} - \frac{c}{d}$ and end up with $\frac{ad - cb}{bd}$. We just notice that subtracting two integers yields an integer and then apply the same reasoning as before.

Discussion (7 minutes)

- Now that we've shown that the set of rational numbers is closed under addition, let's extend our thinking from the realm of numbers to the realm of algebra: Is the set of rational expressions also closed under addition? To help answer this question, let's return to the Opening Exercise.
- In the Opening Exercise, you showed that $\frac{x}{x+2} - \frac{3}{x+1} = \frac{(x^2 + x) - (3x + 6)}{(x+2)(x+1)}$. Is this result a rational expression? We'll need to recall some information about what a rational expression is.

> All rational expressions can be put into the form $\frac{P}{Q}$ where P and Q are polynomial expressions and Q is not the zero polynomial. Rational expressions do not necessarily start out in this form, but all can be rewritten in it.

- Now let's examine the expression $\frac{(x^2+x)-(3x+6)}{(x+2)(x+1)}$. Does this expression meet the above requirement?
  - *Yes, the numerator involves subtracting two polynomials, and the denominator involves multiplying two polynomials, so the quotient is a rational expression.*
- This analysis should give you some idea of what happens in the general case. Our work with rational numbers hinged on our understanding of *integers;* our work with rational expressions hinges on *polynomials*.
- We can make an argument for the closure of rational expressions under addition that closely parallels the argument we made about rational numbers. Work together with a partner to develop an argument to this end.
  - *If x and y are rational expressions, then there are polynomials a, b, c, and d so that $x = \frac{a}{b}$ and $y = \frac{c}{d}$, and neither b nor d is the zero polynomial.*
  - *The sum of x and y is $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.*
  - *The terms ad and bc are polynomials because they are products of polynomials, and polynomials are closed under multiplication.*
  - *The numerator $ad + bc$ is a polynomial because it is the sum of polynomials ad and bc, and polynomials are closed under addition.*
  - *The denominator bd is a polynomial because it is the product of polynomials b and d, and polynomials are closed under multiplication.*
  - *The sum is a rational number because the numerator and denominator are both polynomials.*
- In the case of integers, we know that bd cannot be zero unless either b or d is zero. Similarly, it can be shown that bd cannot be the zero polynomial unless b or d is the zero polynomial. So, the expression $\frac{ad+bc}{bd}$ is a bona fide rational expression after all! We could make a similar argument to show that the set of rational expressions is closed under subtraction also.
- Can you summarize the Discussion so far? Try to convey the main point of the lesson to another student in one or two sentences.
  - *We showed that the set of rational numbers is closed under addition and subtraction, and then we showed that the set of rational expressions is closed under addition and subtraction, too.*
- Let's summarize the logic of the lesson as well: How did we establish closure for the set of rational numbers? How did we establish closure for the set of rational expressions? Make your answers as concise as possible.
  - *We established closure for the set of rational numbers by using closure properties for the set of integers; then, we established closure for the set of rational expressions by using closure properties for the set of polynomials.*
- Now that we have studied the structure of addition and subtraction, let's turn our attention to multiplication and division.

Exercises 3–6 (4 minutes)

These exercises review how to multiply and divide fractions. Students then multiply and divide rational expressions.

3. **Multiply the fractions:** $\frac{2}{5} \cdot \frac{3}{4}$.

   $$\frac{2}{5} \cdot \frac{3}{4} = \frac{6}{20}$$

4. **Divide the fractions:** $\frac{2}{5} \div \frac{3}{4}$.

   $$\frac{2}{5} \div \frac{3}{4} = \frac{2}{5} \cdot \frac{4}{3} = \frac{8}{15}$$

5. **Multiply the expressions:** $\frac{x+1}{x+2} \cdot \frac{3x}{x-4}$.

   $$\frac{x+1}{x+2} \cdot \frac{3x}{x-4} = \frac{(x+1) \cdot 3x}{(x+2)(x-4)}$$

6. **Divide the expressions:** $\frac{x+1}{x+2} \div \frac{3x}{x-4}$.

   $$\frac{x+1}{x+2} \div \frac{3x}{x-4} = \frac{x+1}{x+2} \cdot \frac{x-4}{3x} = \frac{(x+1)(x-4)}{(x+2) \cdot 3x}$$

Scaffolding:

- Consider challenging advanced students to justify the procedure used to divide fractions.
- $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$

Discussion (4 minutes)

- Is the set of rational numbers closed under multiplication? What about the set of rational expressions? Let's explore these questions together.
- Once again, we can sometimes learn more by doing less. Let's reexamine the problem in which you multiplied two fractions but, this time, without doing the arithmetic.
- We have $\frac{2}{5} \cdot \frac{3}{4} = \frac{2 \cdot 3}{5 \cdot 4}$. Is the result a rational number? Why or why not?
  - *Yes. The numerator and the denominator are each the product of two integers, and the denominator is not zero.*
- Does this argument work in the general case? Take a moment to find out.
  - $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$
  - *The numerator and denominator are each integers, and the denominator cannot be zero. This proves that the set of rational numbers is closed under multiplication.*
- Next, let's examine the set of rational expressions. Is this set closed under multiplication? Let's analyze the problem from the exercise set. We showed that $\frac{x+1}{x+2} \cdot \frac{3x}{x-4} = \frac{(x+1) \cdot 3x}{(x+2)(x-4)}$. Is the result a rational expression? Why or why not?
  - *Yes. The numerator is a product of polynomials, and the denominator is the product of nonzero polynomials.*

MP.3

- Does this apply for multiplication of rational expressions in general? Explain.
  - *Yes. The product of rational expressions $x = \frac{a}{b}$ and $y = \frac{c}{d}$ is equal to $\frac{ac}{bd}$. Both ac and bd are polynomials because they are the products of polynomials, and polynomials are closed under multiplication. Also, bd is not the zero polynomial because neither b nor d are zero polynomials. Therefore, $\frac{ac}{bd}$ is a ratio of polynomials, which means that it is a rational expression.*
- Summarize this part of the Discussion in one or two sentences. Share your response with a partner.
  - *Both the set of rational numbers and the set of rational expressions are closed under multiplication.*
- Okay. On to division! Try the following exercises.

Exercises 7–8 (4 minutes)

7. Construct an argument that shows that the set of rational numbers is closed under division. That is, if x and y are rational numbers (with y nonzero) and $w = \frac{x}{y}$, prove that w must also be a rational number.

Let $x = \frac{a}{b}$ and let $y = \frac{c}{d}$, with both b and d nonzero.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

This is indeed a rational number. Thus, the set of rational numbers is closed under division by a nonzero number.

8. How could you modify your argument to show that the set of rational expressions is also closed under division by a nonzero rational expression? Discuss your response with another student.

The only change is that a, b, c, and d represent polynomials rather than integers. The numerator and denominator of the quotient are polynomials because they both represent the product of polynomials, and polynomials are closed under multiplication. This means that the quotient is a ratio of polynomials, which fits our definition of a rational expression.

Closing (2 minutes)

- Use your notebook to briefly summarize what you learned in today's lesson.
  - *The set of rational expressions has a structure similar to the set of rational numbers. In particular, both sets are closed under addition, subtraction, multiplication, and division by a nonzero term. The properties of rational numbers are derived from properties of the integers, whereas the properties of rational expressions are derived from properties of polynomials.*
- If time permits, choose a student to share what she wrote with the class.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 10: The Structure of Rational Expressions

Exit Ticket

1. Payton says that rational expressions are not closed under addition, subtraction, multiplication, or division. His claim is shown below. Is he correct for each case? Justify your answers.

   a. $\dfrac{x}{2x+1}+\dfrac{x+1}{2x+1}=1$, and 1 is a whole number, not a rational expression.

   b. $\dfrac{3x-1}{2x+1}-\dfrac{3x-1}{2x+1}=0$, and 0 is a whole number, not a rational expression.

   c. $\dfrac{x-1}{x+1}\cdot\dfrac{x+1}{1}=x-1$, and $x-1$ is a whole number, not a rational expression.

   d. $\dfrac{x-1}{x+1}\div\dfrac{1}{x+1}=x-1$, and $x-1$ is a whole number, not a rational expression.

2. Simplify the following rational expressions by rewriting them with a single polynomial denominator.

a. $\dfrac{3}{x-1}+\dfrac{2}{x}$

b. $\dfrac{2}{x-2}-\dfrac{3}{x}$

c. $\dfrac{x+1}{x-1}\cdot\dfrac{x}{x-1}$

d. $\dfrac{x+2}{x-1}\div\dfrac{x-2}{x^2-1}$

Exit Ticket Sample Solutions

1. **Payton says that rational expressions are not closed under addition, subtraction, multiplication, or division. His claim is shown below. Is he correct for each case? Justify your answers.**

   a. $\frac{x}{2x+1} + \frac{x+1}{2x+1} = 1$**, and** 1 **is a whole number, not a rational expression.**

   No, he is not correct. $\frac{x}{2x+1} + \frac{x+1}{2x+1} = \frac{2x+1}{2x+1}$ ***The numerator and denominator are both polynomials.***

   b. $\frac{3x-1}{2x+1} - \frac{3x-1}{2x+1} = 0$**, and** 0 **is a whole number, not a rational expression.**

   No, he is not correct. $0 = \frac{0}{1}$ ***The numerator and denominator are both polynomials since integers are an example of polynomials.***

   c. $\frac{x-1}{x+1} \cdot \frac{x+1}{1} = x - 1$**, and** $x - 1$ **is a whole number, not a rational expression.**

   No, he is not correct. $\frac{x-1}{x+1} \cdot \frac{x+1}{x-1} = \frac{x^2-1}{x+1}$ ***The numerator and denominator are both polynomials.***

   d. $\frac{x-1}{x+1} \div \frac{1}{x+1} = x - 1$**, and** $x - 1$ **is a whole number, not a rational expression.**

   No, he is not correct. $\frac{x-1}{x+1} \div \frac{1}{x+1} = \frac{x^2-1}{x+1}$ ***The numerator and denominator are both polynomials.***

2. **Simplify the following rational expressions by rewriting them with a single polynomial denominator.**

   a. $\frac{3}{x-1} + \frac{2}{x}$

   $\frac{5x-2}{x^2-x}$

   b. $\frac{2}{x-2} - \frac{3}{x}$

   $\frac{-x+6}{x^2-2x}$

   c. $\frac{x+1}{x-1} \cdot \frac{x}{x-1}$

   $\frac{x^2+x}{(x-1)^2}$

   d. $\frac{x+2}{x-1} \div \frac{x-2}{x^2-1}$

   $\frac{x^2+3x+2}{x-2}$

Problem Set Sample Solutions

1. Given $\frac{x+1}{x-2}$ and $\frac{x-1}{x^2-4}$, show that performing the following operations results in another rational expression.

   a. Addition

   $$\frac{x+1}{x-2}+\frac{x-1}{x^2-4}=\frac{x^2+3x+2+x-1}{x^2-4}=\frac{x^2+4x+1}{x^2-4}$$

   b. Subtraction

   $$\frac{x+1}{x-2}-\frac{x-1}{x^2-4}=\frac{x^2+3x+2-x+1}{x^2-4}=\frac{x^2+2x+3}{x^2-4}$$

   c. Multiplication

   $$\frac{x+1}{x-2}\cdot\frac{x-1}{x^2-4}=\frac{x^2-1}{(x-2)(x^2-4)}$$

   d. Division

   $$\frac{x+1}{x-2}\div\frac{x-1}{x^2-4}=\frac{x^2+3x+2}{x-1}$$

2. Find two rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ that produce the result $\frac{x-1}{x^2}$ when using the following operations. Answers for each type of operation may vary. Justify your answers.

   a. Addition

   $$\frac{x}{x^2}+\frac{-1}{x^2}=\frac{x-1}{x^2}$$

   b. Subtraction

   $$\frac{x}{x^2}-\frac{1}{x^2}=\frac{x-1}{x^2}$$

   c. Multiplication

   $$\frac{1}{x}\cdot\frac{x-1}{x}=\frac{x-1}{x^2}$$

   d. Division

   $$\frac{1}{x}\div\frac{x}{x-1}=\frac{x-1}{x^2}$$

3. Find two rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ that produce the result $\frac{2x+2}{x^2-x}$ when using the following operations. Answers for each type of operation may vary. Justify your answers.

   a. Addition

   $$\frac{2x}{x^2-x}+\frac{2}{x^2-x}=\frac{2x+2}{x^2-x}$$

   b. Subtraction

   $$\frac{2x}{x^2-x}-\frac{-2}{x^2-x}=\frac{2x+2}{x^2-x}$$

   c. Multiplication

   $$\frac{2}{x}\cdot\frac{x+1}{x-1}=\frac{2x+2}{x^2-x}$$

   d. Division

   $$\frac{2}{x}\div\frac{x-1}{x+1}=\frac{2x+2}{x^2-x}$$

4. Consider the rational expressions A, B and their quotient, $\frac{A}{B}$, where B is not equal to zero.

   a. For some rational expression C, does $\frac{AC}{BC}=\frac{A}{B}$?

   Whenever $C \neq 0$, $\frac{AC}{BC}=\frac{A}{B}$.

   b. Let $A=\frac{x}{y}+\frac{1}{x}$ and $B=\frac{y}{x}+\frac{1}{y}$. What is the least common denominator of every term of each expression?

   xy

   c. Find AC, BC where C is equal to your result in part (b). Then, find $\frac{AC}{BC}$. Simplify your answer.

   $$AC=x^2+y$$
   $$BC=y^2+x$$
   $$\frac{AC}{BC}=\frac{x^2+y}{y^2+x}$$

   d. Express each rational expression A, B as a single rational term, that is, as a division between two polynomials.

   $$A=\frac{x^2+y}{xy}$$
   $$B=\frac{y^2+x}{xy}$$

e. **Write $\frac{A}{B}$ as a multiplication problem.**

$$\frac{A}{B} = A \cdot \frac{1}{B}$$

f. **Use your answers to parts (d) and (e) to simplify $\frac{A}{B}$.**

$$\frac{A}{B} = \frac{x^2 + y}{xy} \cdot \frac{xy}{y^2 + x}$$
$$= \frac{x^2 + y}{y^2 + x}$$

g. **Summarize your findings. Which method do you prefer using to simplify rational expressions?**

We can simplify complex rational expressions by either multiplying both the numerators and denominators by the least common denominator, or we can use the fact that division by a number is multiplication by its reciprocal. Answers may vary on preference.

5. **Simplify the following rational expressions.**

a. $\dfrac{\frac{1}{y} - \frac{1}{x}}{\frac{x}{y} - \frac{y}{x}}$

$$\frac{\frac{1}{y} - \frac{1}{x}}{\frac{x}{y} - \frac{y}{x}} = \frac{\frac{x - y}{xy}}{\frac{x^2 - y^2}{xy}} = \frac{1}{x + y}$$

b. $\dfrac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x^2} - \frac{1}{y^2}}$

$$\frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x^2} - \frac{1}{y^2}} = \frac{1}{\frac{1}{x} - \frac{1}{y}} = \frac{xy}{y - x}$$

c. $\dfrac{\frac{1}{x^4} - \frac{1}{y^2}}{\frac{1}{x^4} + \frac{2}{x^2y} + \frac{1}{y^2}}$

$$\frac{\frac{1}{x^4} - \frac{1}{y^2}}{\frac{1}{x^4} + \frac{2}{x^2y} + \frac{1}{y^2}} = \frac{\left(\frac{1}{x^2} + \frac{1}{y}\right)\left(\frac{1}{x^2} - \frac{1}{y}\right)}{\left(\frac{1}{x^2} + \frac{1}{y}\right)^2} = \frac{\frac{1}{x^2} - \frac{1}{y}}{\frac{1}{x^2} + \frac{1}{y}} = \frac{\frac{y - x^2}{x^2y}}{\frac{y + x^2}{x^2y}} = \frac{y - x^2}{y + x^2}$$

d. $\dfrac{\frac{1}{x-1} - \frac{1}{x}}{\frac{1}{x-1} + \frac{1}{x}}$

$$\frac{\frac{x - x + 1}{(x - 1)x}}{\frac{x + x - 1}{(x - 1)x}} = \frac{1}{2x - 1}$$

6. **Find A and B that make the equation true. Verify your results.**

   a. $\frac{A}{x+1} + \frac{B}{x-1} = \frac{2}{x^2-1}$

$$\frac{A(x-1)+B(x+1)}{(x+1)(x-1)} = \frac{2}{(x+1)(x-1)}$$

Therefore,

$$A(x-1)+B(x+1) = 2.$$

Let $x = 1$, $A = -1$

Let $x = -1$, $B = 1$

$$-\frac{1}{x+1} + \frac{1}{x-1} = \frac{2}{x^2-1}$$

   b. $\frac{A}{x+3} + \frac{B}{x+2} = \frac{2x-1}{x^2+5x+6}$

$$\frac{A(x+2)+B(x+3)}{(x+3)(x+2)} = \frac{2x-1}{(x+3)(x+2)}$$

Therefore,

$$A(x+2)+B(x+3) = 2x-1.$$

Let $x = -3$, $A = 7$

Let $x = -2$, $B = -5$

$$\frac{7}{x+3} - \frac{5}{x+2} = \frac{2x-1}{x^2+5x+6}$$

7. **Find A, B, and C that make the equation true. Verify your result.**

$$\frac{Ax+B}{x^2+1} + \frac{C}{x+2} = \frac{x-1}{(x^2+1)(x+2)}$$

$$\frac{Ax+B}{x^2+1} + \frac{C}{x+2} = \frac{x-1}{(x^2+1)(x+2)}, \quad (Ax+B)(x+2)+C(x^2+1) = x-1,$$

$$Ax^2+2Ax+Bx+2B+Cx^2+C = x-1$$

Therefore,

$$A+C = 0, \quad 2A+B = 1$$

and $2B+C = -1$.

$$A = \frac{3}{5}, \quad B = -\frac{1}{5}, \quad C = -\frac{3}{5}$$

$$\frac{\frac{3}{5}x - \frac{1}{5}}{x^2+1} + \frac{-\frac{3}{5}}{x+2} = \frac{x-1}{(x^2+1)(x+2)}$$

Lesson 11: Rational Functions

Student Outcomes

- Students simplify rational expressions to lowest terms.
- Students determine the domain of rational functions.

Lesson Notes

In Algebra II, students simplified rational expressions to lowest terms and performed arithmetic operations with them, in preparation for solving rational equations. In the previous lesson, students verified that rational expressions are closed under addition, subtraction, multiplication, and division. In this lesson, students first review the concept of equivalent rational expressions from Algebra II Module 1 Lesson 22. They attend to precision (MP.6) in keeping track of the values of the variable that must be excluded from the domain to avoid division by zero. This lesson then introduces rational functions as functions that can be written as quotients of two polynomial functions. Then, students determine whether functions are rational and identify their domain (range is addressed later when they graph rational functions). Reviewing the process of reducing rational expressions to lowest terms prepares students for later lessons in which they graph and compose rational functions.

Classwork

Opening Exercise (4 minutes)

The Opening Exercise gets students thinking about factoring polynomial expressions, which is a skill they need to complete their work with rational functions in this and subsequent lessons. Students should complete this exercise independently. After a few minutes, select students to share their solutions. The factored expressions could also be written on individual white boards for quick checks.

Scaffolding:

- Cue students to look for patterns that can help them factor the expressions, for example, difference of squares, common factors, or the binomial theorem.
- Ask students to consider simpler examples, such as $x^2 - 9 = (x-3)(x+3)$; $y^2 + 2y - 15 = (y-3)(y+5)$; $a^2 + 5a + 4 = (a+4)(a+1)$.

Opening Exercise

Factor each expression completely:

a. $9x^4 - 16x^2$

$x^2(3x+4)(3x-4)$

b. $2x^3 + 5x^2 - 8x - 20$

$(x^2-4)(2x+5) = (x+2)(x-2)(2x+5)$

c. $x^3 + 3x^2 + 3x + 1$

$(x+1)^3$

d. $8x^3 - 1$

$(2x-1)(4x^2+2x+1)$

Discussion (5 minutes): Equivalent Rational Expressions

In Algebra II Module 1 Lesson 22, students practiced reducing rational expressions to lowest terms, taking care to note values of the variable that must be excluded to avoid division by zero. In this lesson, this idea is extended to finding the domain of a rational function. Use this Discussion to reactivate students' knowledge of rational expressions and reducing a rational expression to lowest terms.

- Recall that in Algebra II and the previous lesson, we described rational expressions as expressions that can be put into the form $\frac{P}{Q}$ where P and Q are polynomial expressions and Q is not the zero polynomial. For example, $\frac{x}{x^2 - 3x + 2}$, $x^2 + 1$, $\frac{x^3 - 1}{x^2 + 2}$, 0, and $1 + \frac{3}{x}$ are all rational expressions.
- What does it mean for two rational expressions to be equivalent?
  - *That although the expressions may be in different forms, each expression takes on the same value for any value of the variables. That is, if we substitute a value such as 3 for x into each expression, the values of the expressions are the same.*
- Notice that $\frac{x}{x^3 + x} = \frac{x}{x(x^2 + 1)}$. Are $\frac{x}{x^3 + x}$ and $\frac{1}{x^2 + 1}$ equivalent rational expressions?
  - *No. The first expression is undefined for $x = 0$, but the second is defined for all values of x. Thus, they are not equivalent expressions.*
- What should we do to make these equivalent expressions?
  - *Excluding the value of 0 from the set of possible values of x makes both expressions equivalent because $\frac{x}{x^3 + x} = \frac{1}{x^2 + 1}$ only for $x \neq 0$.*
- What does it mean to simplify a rational expression to lowest terms?
  - *We divide any common factors from the numerator and denominator, leaving polynomials of the lowest possible degree.*

MP.6
- What do we need to pay attention to in order to ensure that, as we simplify a rational expression to lowest terms, we ensure that the resulting rational expressions are equivalent to the original one?
  - *We need to exclude any value of the variable that caused division by zero in the original expression.*

Example 1 (5 minutes)

This example provides a review of reducing rational expressions to lowest terms from Algebra II Module 1. It is important to emphasize excluding the value $x = 3$ from the possible values for x as the expression is simplified.

Example 1

Simplify the expression $\frac{x^2 - 5x + 6}{x - 3}$ to lowest terms, and identify the value(s) of x that must be excluded to avoid division by zero.

Give students time to think about how to approach this task before leading them through a solution.

> ***Since $x^2 - 5x + 6 = (x-2)(x-3)$, our original expression can be written as $\frac{x^2-5x+6}{x-3} = \frac{(x-2)(x-3)}{x-3}$. To simplify this expression to lowest terms, we need to divide the numerator and denominator by any common factors. The only common factor in this example is $x - 3$. However, we can only divide by $x - 3$ if $x - 3 \neq 0$, which means that we have to exclude 3 as a possible value of x.***
>
> ***Thus, if $x \neq 3$, we have***
>
> $$\frac{x^2-5x+6}{x-3} = \frac{(x-2)(x-3)}{x-3} =$$
> $$= \frac{(x-2)(x-3)}{x-3} \cdot \frac{\frac{1}{x-3}}{\frac{1}{x-3}}$$
> $$= x - 2.$$
>
> ***So, as long as $x \neq 3$, the expressions $\frac{x^2-5x+6}{x-3}$ and $x - 2$ are equivalent.***

Exercise 1 (6 minutes): Simplifying Rational Expressions to Lowest Terms

> **Exercise 1: Simplifying Rational Expressions to Lowest Terms**
>
> **1. Simplify each rational expression to lowest terms, specifying the values of x that must be excluded to avoid division by zero.**
>
> **a.** $\frac{x^2-6x+5}{x^2-3x-10}$
>
> ***The denominator factors into $x^2 - 3x - 10 = (x-5)(x+2)$, so to avoid division by zero, we must have $x \neq 5$ and $x \neq -2$. Thus, $\frac{x^2-6x+5}{x^2-3x-10} = \frac{(x-5)(x-1)}{(x-5)(x+2)} = \frac{x-1}{x+2}$, where $x \neq -2$ and $x \neq 5$.***
>
> **b.** $\frac{x^3+3x^2+3x+1}{x^3+2x^2+x}$
>
> ***The denominator factors into $x^3 + 2x^2 + x = x(x+1)^2$, so to avoid division by zero, we must have $x \neq 0$ and $x \neq -1$. Thus, $\frac{x^3+3x^2+3x+1}{x^3+2x^2+x} = \frac{(x+1)^3}{x(x+1)^2} = \frac{x+1}{x}$, where $x \neq 0$ and $x \neq 1$.***
>
> **c.** $\frac{x^2-16}{x^2+2x-8}$
>
> ***The denominator factors into $x^2 + 2x - 8 = (x-2)(x+4)$, so to avoid division by zero, we must have $x \neq 2$ and $x \neq -4$. Thus, $\frac{x^2-16}{x^2+2x-8} = \frac{(x-4)(x+4)}{(x-2)(x+4)} = \frac{x-4}{x-2}$, where $x \neq 2$ and $x \neq -4$.***
>
> **d.** $\frac{x^2-3x-10}{x^3+6x^2+12x+8}$
>
> ***The denominator factors into $x^3 + 6x^2 + 12x + 8 = (x+2)^3$, so to avoid division by zero, we must have $x \neq -2$. Thus, $\frac{x^2-3x-10}{x^3+6x^2+12x+8} = \frac{(x-5)(x+2)}{(x+2)^3} = \frac{x-5}{(x+2)^2}$, where $x \neq -2$.***

e. $\dfrac{x^3+1}{x^2+1}$

While $x^3 + 1 = (x+1)(x^2 - x + 1)$, the polynomial expression in the denominator does not factor. Thus, this expression is already simplified to lowest terms. Since $x^2 + 1 \geq 1$ for all values of x, the denominator is never zero. Thus, there are no values of x that need to be excluded.

Discussion (5 minutes): Identifying Rational Functions

This Discussion describes rational functions as those that can be written as the quotient of two polynomial functions. Continue to emphasize the domain of a rational function through this Discussion and throughout the lesson.

- We are now interested in using rational expressions to define functions.
- Remember that a function $f: X \to Y$ is a correspondence between two sets X and Y. To specify a function, we need to know its domain and the rule used to match elements of X to elements of Y. We now want to define functions whose rule of assignment can be described using rational expressions.
- A rational function is a function whose rule of assignment can be written in the form $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomial functions and Q is not the zero polynomial. What can you recall about the structure of a polynomial function?
  - *It can be written in the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots a_1 x + a_0$, where $a_n, a_{n-1}, \ldots, a_0$ are real numbers and n is a whole number.*
- Is the function $f(x) = \dfrac{x^2 + 5x + 4}{x^2 - 16}$ a rational function? Explain.
  - *Yes. The numerator and denominator of f are both polynomial functions.*
- Let's see if we can use our definition to classify some more complicated functions. Which of the functions shown here are rational functions? Explain how you know.

$$f(x) = \frac{5x^3 - 6x + 2}{\pi x^2}$$

$$g(x) = \frac{x^{200}}{x^{200} - 1}$$

$$h(x) = \sqrt{2x+1} - 2$$

$$j(x) = 17$$

$$k(x) = \frac{\cos(x)}{x^2 + 1}$$

  - *Both f and g are rational functions because both the numerator and denominator of each function are polynomials; for example, the terms have real-numbered coefficients and powers of x that are integers. The function h is not rational because it cannot be written as a quotient of polynomial functions. Function j is a rational function with numerator $P(x) = 17$ and denominator $Q(x) = 1$, and k is not a rational function because $P(x) = \cos(x)$ cannot be written as a polynomial function.*

Example 2 (5 minutes)

This example demonstrates how the rule of a rational function can be expressed in an equivalent form by dividing the numerator and denominator by common factors and explicitly stating a restricted domain. The exercise should be completed in pairs and, after a few minutes, the responses should be reviewed as part of a teacher-led discussion. Alternatively, the example could be completed as part of a teacher-led discussion.

- How can we simplify the expression of the function $f(x) = \frac{x^2+5x+4}{x^2-16}$?
  - *Factor the numerator and denominator, and look for common factors that can be divided out.*
- What characteristics of the denominator could help us to factor it?
  - *The presence of a difference of squares in the denominator could help us when we factor the denominator.*
- Why can't we rewrite the equation for f as $f(x) = \frac{x+1}{x-4}$?
  - *Without indicating the restricted values on the domain of the function, there is no way to tell from the simplified expression that the function is undefined at $x = 4$. Thus, if we don't explicitly identify the additional restriction $x \neq 4$ on the domain, we don't have the same function.*
- And how do we know that 4 and -4 are restricted values not in the domain of f?
  - *The denominator of the function is 0 for each of these values of x, which results in the function being undefined.*
- Remember that a function is a rule and a domain, so if we change the domain, we have substantially changed the function. When we simplify the expression that defines a rational function, how can we make sure that we do not change the domain?
  - *We can make sure not change the domain by identifying the restricted values from the factored form of the original function before it is simplified to lowest terms. The restricted values represent those numbers that, when substituted into the function, produce a fraction with a denominator equal to 0.*
- And how can we write a rational function so that its expression has been simplified to lowest terms and the restricted domain values are indicated?
  - *The simplified expression of the function can be written along with an explicit statement of the excluded values of the variable.*

Scaffolding:

Cue students to look for a binomial pattern to help them factor the numerator of f.

MP.7

Example 2

Let $f(x) = \frac{2x^4+6x^3+6x^2+2x}{3x^2+3x}$. Simplify the rational expression $\frac{2x^4+6x^3+6x^2+2x}{3x^2+3x}$ to lowest terms, and use the simplified form to express the rule of f. Be sure to indicate any restrictions on the domain.

$$\frac{2x^4+6x^3+6x^2+2x}{3x^2+3x} = \frac{2x(x^3+3x^2+3x+1)}{3x(x+1)} = \frac{2x(x+1)^3}{3x(x+1)} = \frac{2(x+1)^2}{3} \text{ if } x \neq -1 \text{ and } x \neq 0.$$

Then, $f(x) = \frac{2(x+1)^2}{3}$ *for* $x \neq 0$ *and* $x \neq -1$.

Exercise 2 (7 minutes)

Have students complete this exercise in pairs. After a few minutes, select students to share their responses. If personal white boards are available, students could write their answers on the boards for quick checks.

> *Scaffolding:*
> Have advanced students form conjectures about the range of each function.

MP.6

Exercise 2

2. Determine the domain of each rational function, and express the rule for each function in an equivalent form in lowest terms.

a. $f(x) = \frac{(x+2)^2(x-3)(x+1)}{(x+2)(x+1)}$

The domain is all real numbers x so that $x \neq -1$ and $x \neq -2$.

$f(x) = \frac{(x+2)^2(x-3)(x+1)}{(x+2)(x+1)} = (x+2)(x-3)$ ***for*** $x \neq -1$ ***and*** $x \neq -2$

b. $f(x) = \frac{x^2-6x+9}{x-3}$

The domain is all real numbers x so that $x \neq 3$.

$f(x) = \frac{(x-3)^2}{x-3} = x-3$ ***for*** $x \neq 3$

c. $f(x) = \frac{3x^3-75x}{x^3+15x^2+75x+125}$

The domain is all real numbers x so that $x \neq -5$.

$f(x) = \frac{3x(x^2-25)}{(x+5)^3} = \frac{3x(x+5)(x-5)}{(x+5)^3} = \frac{3x(x-5)}{(x+5)^2}$ ***for*** $x \neq -5$

Closing (3 minutes)

Have students reflect on the questions below. After a minute, ask them to share their thoughts with a partner.

- How do we identify the domain of a rational function?
  - *A rational function has the domain of all real numbers except for any value of x that causes division by zero.*
- Explain why the functions $f(x) = \frac{x}{x-3}$ and $g(x) = \frac{x(x-1)}{(x-1)(x-3)}$ are not the same function.
  - *The first function $f(x) = \frac{x}{x-3}$ is defined for all $x \neq 3$, but the second function $g(x) = \frac{x(x-1)}{(x-1)(x-3)}$ is defined for $x \neq 3$ and $x \neq 1$. Since the two functions do not agree for every value of x, they are not the same function.*
  - *The two functions have different domains, so they are not the same function.*

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 11: Rational Functions

Exit Ticket

1. Identify whether the functions shown are rational:

   a. $f(x) = \dfrac{x}{x^2 + 1}$

   b. $f(x) = \dfrac{\sqrt{x}}{x^2 + 1}$

   c. $f(x) = \dfrac{x}{x^{0.4} + 1}$

   d. $f(x) = \left(\dfrac{x}{x^2 + 1}\right)^2$

   e. $f(x) = \dfrac{\sqrt{2}x}{ex^2 + \sqrt{\pi}}$

2. Anmol says $f(x) = \dfrac{x+1}{x^2 - 1}$ and $g(x) = \dfrac{1}{x-1}$ represent the same function. Is she correct? Justify your answer.

Exit Ticket Sample Solutions

1. **Identify whether the functions shown are rational:**

   a. $f(x) = \frac{x}{x^2+1}$

   Yes. Both $P(x) = x$ and $Q(x) = x^2 + 1$ are polynomial functions.

   b. $f(x) = \frac{\sqrt{x}}{x^2+1}$

   No. The function $P(x) = \sqrt{x}$ is not a polynomial function.

   c. $f(x) = \frac{x}{x^{0.4}+1}$

   No. The function $Q(x) = x^{0.4} + 1$ is not a polynomial function.

   d. $f(x) = \left(\frac{x}{x^2+1}\right)^2$

   Yes. When multiplied out, $f(x) = \frac{x^2}{x^4+2x^2+1}$, so f is the quotient of two polynomial functions.

   e. $f(x) = \frac{\sqrt{2}x}{ex^2+\sqrt{\pi}}$

   Yes. While the coefficients are not integers, $P(x) = \sqrt{2}x$ and $Q(x) = ex^2 + \sqrt{\pi}$ are both polynomial functions since all the powers of x are whole numbers.

2. **Anmol says $f(x) = \frac{x+1}{x^2-1}$ and $g(x) = \frac{1}{x-1}$ represent the same function. Is she correct? Justify your answer.**

   She is not correct.

   The function $f(x) = \frac{x+1}{x^2-1}$ is not defined for $x = 1$ and $x = -1$. However, the function $g(x) = \frac{1}{x-1}$ is not defined for $x = 1$. These two functions do not have the same domain, so they are not the same function.

Problem Set Sample Solutions

1. **For each pair of functions f and g, find the domain of f and the domain of g. Indicate whether f and g are the same function.**

   a. $f(x) = \frac{x^2}{x}$, $g(x) = x$

   The domain of f is all real numbers x with $x \neq 0$. The domain of g is all real numbers x.

   No, functions f and g are not the same function.

b. $f(x) = \frac{x}{x}$, $g(x) = 1$

The domain of f is all real numbers x with $x \neq 0$. The domain of g is all real numbers x.

No, functions f and g are not the same function.

c. $f(x) = \frac{2x^2 + 6x + 8}{2}$, $g(x) = x^2 + 6x + 8$

The domain of f is all real numbers x. The domain of g is all real numbers x.

Yes, functions f and g are the same function.

d. $f(x) = \frac{x^2 + 3x + 2}{x + 2}$, $g(x) = x + 1$

The domain of f is all real numbers x with $x \neq -2$. The domain of g is all real numbers x.

No, functions f and g are not the same function.

e. $f(x) = \frac{x + 2}{x^2 + 3x + 2}$, $g(x) = \frac{1}{x + 1}$

The domain of f is all real numbers x with $x \neq -2$ and $x \neq -1$. The domain of g is all real numbers x with $x \neq -1$.

No, functions f and g are not the same function.

f. $f(x) = \frac{x^4 - 1}{x^2 - 1}$, $g(x) = x^2 + 1$

The domain of f is all real numbers x with $x \neq 1$ and $x \neq -1$. The domain of g is all real numbers x.

No, functions f and g are not the same function.

g. $f(x) = \frac{x^4 - 1}{x^2 + 1}$, $g(x) = x^2 - 1$

Because $x^2 + 1$ is never zero, the domain of f is all real numbers x. The domain of g is all real numbers x.

Yes, functions f and g are the same function.

h. $f(x) = \frac{x^4 - x}{x^2 + x}$, $g(x) = \frac{x^3 - 1}{x + 1}$

The domain of f is all real numbers x with $x \neq 0$ and $x \neq -1$. The domain of g is all real numbers x with $x \neq -1$.

No, functions f and g are not the same function.

i. $f(x) = \frac{x^4 + x^3 + x^2}{x^2 + x + 1}$, $g(x) = x^2$

Because $x^2 + x + 1$ doesn't factor, the denominator of f is never zero, and the domain of f is all real numbers x. The domain of g is also all real numbers x.

Yes, functions f and g are the same function.

2. **Determine the domain of each rational function, and express the rule for each function in an equivalent form in lowest terms.**

a. $f(x) = \frac{x^4}{x^2}$

The domain of f is all real numbers x with $x \neq 0$.

$f(x) = \frac{x^4}{x^2} = x^2$, *where* $x \neq 0$

b. $f(x) = \frac{3x+3}{15x-6}$

Because $15x - 6 = 3(5x-2)$, *the domain of f is all real numbers x with* $x \neq \frac{2}{5}$.

$f(x) = \frac{3(x+1)}{3(5x-2)} = \frac{x+1}{5x-2}$, *where* $x \neq \frac{2}{5}$

c. $f(x) = \frac{x^2-x-2}{x^2+x}$

Because $x^2 + x = x(x+1)$, *the domain of f is all real numbers x with* $x \neq 0$ *and* $x \neq -1$.

$f(x) = \frac{x^2-x-2}{x^2+x} = \frac{(x-2)(x+1)}{x(x+1)} = \frac{x-2}{x}$, *where* $x \neq 0$ *and* $x \neq -1$

d. $f(x) = \frac{8x^2+2x-15}{4x^2-4x-15}$

Because $4x^2 - 4x - 15 = (2x+3)(2x-5)$, *the domain of f is all real numbers x with* $x \neq -\frac{3}{2}$ *and* $x \neq \frac{5}{2}$.

$f(x) = \frac{8x^2+2x-15}{4x^2-4x-15} = \frac{(2x+3)(4x-5)}{(2x+3)(2x-5)} = \frac{(4x-5)}{(2x-5)}$, *where* $x \neq \frac{5}{2}$ *and* $x \neq -\frac{3}{2}$

e. $f(x) = \frac{2x^3-3x^2-2x+3}{x^3-x}$

Because $x^3 - x = x(x-1)(x+1)$, *the domain of f is all real numbers x with* $x \neq 0$, $x \neq 1$ *and* $x \neq -1$.

$f(x) = \frac{2x^3-3x^2-2x+3}{x^3-x} = \frac{(2x-3)(x^2-1)}{x(x^2-1)} = \frac{2x-3}{x}$, *where* $x \neq 0$ *and* $x \neq \pm 1$

f. $f(x) = \frac{3x^3+x^2+3x+1}{x^3+x}$

Because $x^3 + x = x(x^2+1)$, *the domain of f is all real numbers x with* $x \neq 0$.

$f(x) = \frac{3x^3+x^2+3x+1}{x^3+x} = \frac{(3x+1)(x^2+1)}{x(x^2+1)} = \frac{3x+1}{x}$, *where* $x \neq 0$

3. **For each pair of functions below, calculate $f(x)+g(x)$, $f(x)-g(x)$, $f(x)\cdot g(x)$, and $\frac{f(x)}{g(x)}$. Indicate restrictions on the domain of the resulting functions.**

a. $f(x)=\frac{2}{x},\ g(x)=\frac{x}{x+2}$

$$f(x)+g(x)=\frac{2}{x}+\frac{x}{x+2}=\frac{2(x+2)}{x(x+2)}+\frac{x(x)}{(x+2)(x)}=\frac{x^2+2x+4}{x(x+2)}, \text{ where } x\neq 0,\ -2$$

$$f(x)-g(x)=\frac{2}{x}-\frac{x}{x+2}=\frac{2(x+2)}{x(x+2)}-\frac{x(x)}{(x+2)(x)}=\frac{-x^2+2x+4}{x(x+2)}, \text{ where } x\neq 0,\ -2$$

$$f(x)\cdot g(x)=\frac{2}{x}\cdot\frac{x}{x+2}=\frac{2x}{x(x+2)}=\frac{2}{(x+2)}, \text{ where } x\neq 0,\ -2$$

$$\frac{f(x)}{g(x)}=\frac{2}{x}\div\frac{x}{x+2}=\frac{2}{x}\cdot\frac{x+2}{x}=\frac{2x+4}{x^2}, \text{ where } x\neq 0,\ -2$$

b. $f(x)=\frac{3}{x+1},\ g(x)=\frac{x}{x^3+1}$

$$f(x)+g(x)=\frac{3}{x+1}+\frac{x}{x^3+1}=\frac{3(x^2-x+1)}{(x+1)(x^2-x+1)}+\frac{x}{x^3+1}=\frac{3x^2-2x+3}{x^3+1}, \text{ where } x\neq -1$$

$$f(x)-g(x)=\frac{3}{x+1}-\frac{x}{x^3+1}=\frac{3(x^2-x+1)}{(x+1)(x^2-x+1)}-\frac{x}{x^3+1}=\frac{3x^2-4x+3}{x^3+1}, \text{ where } x\neq -1$$

$$f(x)\cdot g(x)=\frac{3}{x+1}\cdot\frac{x}{x^3+1}=\frac{3x}{(x+1)(x^3+1)}, \text{ where } x\neq -1$$

$$\frac{f(x)}{g(x)}=\frac{3}{x+1}\div\frac{x}{x^3+1}=\frac{3}{(x+1)}\cdot\frac{x^3+1}{x}=\frac{3(x+1)(x^2-x+1)}{x(x+1)}=\frac{3(x^2-x+1)}{x}, \text{ where } x\neq -1,\ 0$$

Lesson 12: End Behavior of Rational Functions

Student Outcomes

- Students describe the end behavior of rational functions.

Lesson Notes

This lesson offers students opportunities to use tables to analyze the end behavior of rational functions and the behavior of rational functions as they approach restricted input values. This prepares students for subsequent lessons in which they graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior (**F-IF.C.7d**).

Classwork

Opening Exercise (3 minutes)

The work in Algebra II showed students how to analyze the end behavior of polynomials. This lesson begins with a set of exercises that provides an opportunity to recall those skills, and then the end behavior of rational functions is analyzed.

Opening Exercise

Analyze the end behavior of each function below. Then, choose one of the functions, and explain how you determined the end behavior.

a. $f(x) = x^4$

As $x \to \infty$, $f(x) \to \infty$. As $x \to -\infty$, $f(x) \to \infty$.

We have $2^4 = 16$, and $3^4 = 81$. In general, as we use larger and larger inputs, this function produces larger and larger outputs, exceeding all bounds.

f is an even function, so the same remarks apply to negative inputs.

b. $g(x) = -x^4$

As $x \to \infty$, $g(x) \to -\infty$. As $x \to -\infty$, $g(x) \to -\infty$.

We have $-2^4 = -16$, and $-3^4 = -81$. In general, as we use larger and larger inputs, this function produces larger and larger negative outputs, exceeding all bounds.

g is an even function, so the same remarks apply to negative inputs.

c. $h(x) = x^3$

As $x \to \infty$, $h(x) \to \infty$. As $x \to -\infty$, $h(x) \to -\infty$.

We have $2^3 = 8$, and $3^3 = 27$. In general, as we use larger and larger inputs, this function produces larger and larger outputs, exceeding all bounds.

We also have $(-2)^3 = -8$, and $(-3)^3 = -27$. In general, as we use lesser and lesser inputs, this function produces larger and larger negative outputs, exceeding all bounds.

Scaffolding:

- As necessary, remind students that analyzing the end behavior of a function entails finding what value $f(x)$ approaches as $x \to \infty$ or as $x \to -\infty$.
- Also consider showing students the graphs of $f(x) = x^2$ and $g(x) = x^3$ and then asking them to describe the end behavior verbally.

d. $k(x) = -x^3$

As $x \to \infty$, $k(x) \to -\infty$. As $x \to -\infty$, $k(x) \to \infty$.

We have $-2^3 = -8$, and $-3^3 = -27$. In general, as we use larger and larger inputs, this function produces larger and larger negative outputs, exceeding all bounds.

We also have $-(-2)^3 = 8$, and $-(-3)^3 = 27$. In general, as we use lesser and lesser inputs, this function produces larger and larger outputs, exceeding all bounds.

Discussion (3 minutes): Power Functions

The following Discussion builds on the material developed in the Opening Exercises. Students analyze simple **power functions** $g(x) = a \cdot x^n$. This analysis is essential to understanding the end behavior of polynomials as well as that of rational functions.

- Let's take a close look at one of these functions to make sure the reasoning is clear. For $h(x) = x^3$, do you think the outputs grow without bound? For example, do the outputs ever exceed one trillion? Think about this question for a moment, and then share your response with a partner. Try to be as specific as possible.
  - *One trillion is 10^{12}. If we take $x = 10^4$ as an input, we get $f(10^4) = (10^4)^3 = 10^{12}$. So, any input larger than 10^4 produces an output that is larger than one trillion.*
- Good. Now let's focus on the sign of the output. Take $f(x) = x^{13}$ and $g(x) = x^{14}$. When $x = -1000$, is the output positive or negative? Explain your thinking to a partner, being as specific as possible about how you reached your conclusion.
  - *Let's consider $f(-1000) = (-1000)^{13}$. The expression $(-1000)^{13}$ represents the product of 13 negative numbers. Each pair of negative numbers has a positive product, but since there is an odd number of factors, the final product must be negative.*
  - *Now let's consider $g(-1000) = (-1000)^{14}$. This time, there are 7 pairs of negative numbers, each of which has a positive product. Therefore, the final product must be positive.*

Discussion (8 minutes): Reciprocals of Power Functions

- We see that analyzing the end behavior of a power function is straightforward. Now let's turn our attention to some very simple rational functions.
- What do you know about the function $f(x) = \frac{1}{x}$? Give several examples of input-output pairs associated with this function, and then share them with a partner.
  - *This function takes a number and returns its reciprocal. For example, $f(2) = \frac{1}{2}$, $f(10) = \frac{1}{10}$, $f(-3) = -\frac{1}{3}$, $f(1) = 1$, and $f\left(\frac{1}{5}\right) = 5$. At $x = 0$, f is undefined.*
- Describe the end behavior for $f(x) = \frac{1}{x}$.
  - *As $x \to \infty$, $f(x) \to 0$, and as $x \to -\infty$, $f(x) \to 0$.*
- Can you confirm this using a table? Organize several input-output pairs in a table to confirm the behavior of the graph of $f(x) = \frac{1}{x}$ as x approaches 0 from the positive side and as $x \to \infty$.

x	0.001	0.01	0.1	1	10	100	$1{,}000$
$f(x)$	$\frac{1}{0.001} = 1000$	$\frac{1}{0.01} = 100$	$\frac{1}{0.1} = 10$	$\frac{1}{1} = 1$	$\frac{1}{10} = 0.1$	$\frac{1}{100} = 0.01$	$\frac{1}{1000} = 0.001$

- What does the table indicate about $f(x) = \frac{1}{x}$ as $x \to \infty$ and as $x \to 0$ from the positive side?
  - *It approaches 0 as $x \to \infty$, and it approaches ∞ as $x \to 0$ from the positive side.*
- Now let's examine the behavior of the function when the input is a negative number.

x	-0.001	-0.01	-0.1	-1	-10	-100	$-1{,}000$
$f(x)$	$\frac{1}{-0.001} = -1000$	$\frac{1}{-0.01} = -100$	$\frac{1}{-0.1} = -10$	$\frac{1}{-1} = -1$	$\frac{1}{-10} = -0.1$	$\frac{1}{-100} = -0.01$	$\frac{1}{-1000} = -0.001$

- What does the table indicate about $f(x) = \frac{1}{x}$ as $x \to -\infty$ and as $x \to 0$ from the negative side?
  - *It approaches 0 as $x \to -\infty$, and it approaches $-\infty$ as $x \to 0$ from the negative side.*
- What pattern do you notice between the tables?
  - *It appears that $f(-x) = -f(x)$.*
- What type of function displays the type of pattern shown in the tables?
  - *Odd functions*
- How can you prove that f is odd?
  - *f is odd if $f(-x) = -f(x)$. We have $f(-x) = \frac{1}{-x}$ and $-f(x) = -\frac{1}{x}$. These expressions are indeed equivalent, so f is odd.*
- Good. Now let's try analyzing $g(x) = \frac{1}{x^2}$. What can we say about the end behavior of this function? How does the function behave as x approaches 0 from the right and left sides?

> *Scaffolding:*
> - Recall that even functions such as $g(x) = \frac{1}{x^2}$ exhibit symmetry with respect to the y-axis.
> - Odd functions such as $h(x) = \frac{1}{x^3}$ exhibit symmetry with respect to the origin.

x	-100	-10	-1	$\frac{-1}{10}$	$\frac{-1}{100}$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100
$g(x)$	$\frac{1}{10{,}000}$	$\frac{1}{100}$	1	100	$10{,}000$	$10{,}000$	100	1	$\frac{1}{100}$	$\frac{1}{10{,}000}$

- What does the table indicate about $g(x) = \frac{1}{x^2}$ as $x \to \infty$ and as $x \to 0$ from the positive side?
  - *It approaches 0 as $x \to \infty$, and it approaches ∞ as $x \to 0$ from the positive side.*
- What does the table indicate about $g(x) = \frac{1}{x^2}$ as $x \to -\infty$ and as $x \to 0$ from the negative side?
  - *It approaches 0 as $x \to -\infty$, and it approaches ∞ as $x \to 0$ from the negative side.*

- What pattern do you notice between the tables?
  - *It appears that there is symmetry.*
- What type of function displays the type of symmetry shown in the tables?
  - *Even functions*
- How can you prove that $g(x) = \frac{1}{x^2}$ is even?
  - *A function is even if $g(-x) = g(x)$. We have $g(-x) = \frac{1}{(-x)^2}$, and $g(x) = \frac{1}{x^2}$. These expressions are equivalent, so g is even.*
- Now let's analyze the end behavior of $f(x) = \frac{1}{x^3}$ and $g(x) = \frac{1}{x^4}$.
  - *As $x \to \infty$, $f(x) \to 0$, and as $x \to -\infty$, $f(x) \to 0$.*
  - *As $x \to \infty$, $g(x) \to 0$, and as $x \to -\infty$, $g(x) \to 0$.*
- What about the behavior of $f(x)$ and $g(x)$ as the functions approach 0 from the right and left sides?
  - *$f(x) \to \infty$ as $x \to 0$ from the right side.*
  - *$f(x) \to -\infty$ as $x \to 0$ from the left side.*
  - *$g(x) \to \infty$ as $x \to 0$ from the right side.*
  - *$g(x) \to \infty$ as $x \to 0$ from the left side.*

Scaffolding:

- Recall that even functions such as $g(x) = \frac{1}{x^2}$ exhibit symmetry with respect to the y-axis.
- Odd functions such as $h(x) = \frac{1}{x^3}$ exhibit symmetry with respect to the origin.

MP.7 & MP.8

- Can you see how to generalize your results?
  - *If n is any whole number, then as $x \to \infty$, $\frac{1}{x^n} \to 0$, and as $x \to -\infty$, $\frac{1}{x^n} \to 0$.*
  - *If n is any odd whole number, then as $x \to 0$ from the positive side, $\frac{1}{x^n} \to \infty$, and as $x \to 0$ from the negative side, $\frac{1}{x^n} \to -\infty$.*
  - *If n is any even whole number, then as $x \to 0$ from the positive side, $\frac{1}{x^n} \to \infty$, and as $x \to 0$ from the negative side, $\frac{1}{x^n} \to \infty$.*

Discussion: General Polynomials (5 minutes)

- So far, we've examined only the simplest rational functions $f(x) = \frac{1}{x^n}$. What can we say about the end behavior of a more complex rational function such as $f(x) = \frac{5x^3 - 2x^2 + 4x - 16}{2x^3 + 10x^2 - x + 4}$? To answer this question, we need to review some principles regarding the end behavior of polynomials.

MP.3

- Let $g(x) = 5x^3 - 2x^2 + 4x - 16$ and $h(x) = 2x^3 + 10x^2 - x + 4$. Explain why the functions have similar end behavior.
  - *The functions are polynomials of the same degree, and both have positive leading coefficients.*

- We saw in the Opening Exercises that functions of the form $y = x^n$ are simple to analyze, but recall that things get a little more complicated when there are several terms involved. For instance, consider $f(x) = x^3 - 5x^2 - 3x - 1$. See what you can learn by computing $f(4)$ and $f(6)$ term by term. You may use a calculator for this purpose.
  - $f(4) = 64 - 80 - 12 - 1 = -29$
  - $f(6) = 216 - 180 - 18 - 1 = 17$
- In the case where $x = 4$, the second term is larger than the first, and so you ended up subtracting more than you started with, making the answer negative. But in the case where $x = 6$, the first term was large enough that the result was still positive even after performing three subtractions.
- More than anything else, it's the size of the exponent that matters in this analysis. You may recall that the end behavior of a polynomial is determined by the term with the largest exponent. For instance, in the example above, f has the same end behavior as the simple power function $g(x) = x^3$. Do you recall how to demonstrate this formally? Try factoring out x^3 from each term of f to see why its end behavior is similar to g.
  - $f(x) = x^3 - 5x^2 - 3x - 1 = x^3 \cdot (1 - \frac{5}{x} - \frac{3}{x^2} - \frac{1}{x^3})$
- Use this new form for f to compare $f(10^4)$ and $g(10^4)$.
  - $g(10^4) = (10^4)^3 = 10^{12}$
  - $f(10^4) = 10^{12} \cdot \left(1 - \frac{5}{10^4} - \frac{3}{10^8} - \frac{1}{10^{12}}\right)$
- Notice that for this large input, the output of f is about 99.9% of the value of g. If you use an even larger input, the relative value of f grows even closer to that of g. Without actually doing the calculations, can you see intuitively why this is the case? Explain your thinking to a neighbor.
  - *Except for the first term, the terms in parentheses are extremely small, so the multiplier has a value that is very close to* 1. *This explains why the relative values of the two functions are very close when the input is a large number.*
- Let's examine each component in the expression $x^3 \cdot \left(1 - \frac{5}{x} - \frac{3}{x^2} - \frac{1}{x^3}\right)$. What happens when $x \to \infty$?
  - *As* $x \to \infty$, $x^3 \to \infty$, $1 \to 1$, $\frac{5}{x} \to 0$, $\frac{3}{x^2} \to 0$, *and* $\frac{1}{x^3} \to 0$.
  - *Thus,* $1 - \frac{5}{x} - \frac{3}{x^2} - \frac{1}{x^3} \to 1 - 0 - 0 - 0 = 1$.
- This confirms our observation that f and g have similar end behavior. We can say that the function $g(x) = x^3$ is an end-behavior model for $f(x) = x^3 - 5x^2 - 3x - 1$.
- As it happens, knowing how to analyze the end behavior of a polynomial is really all that is needed to understand the end behavior of a rational function. Let's look at some examples.

Example 1 (6 minutes)

In this example, students use what they know about polynomials to analyze a rational function.

MP.3

- Based on our reasoning with f and g above, what would be an end-behavior model for $f(x) = x^4 - 2x^2 + 6x - 3$?
  - $g(x) = x^4$

- Consider the function $f(x) = \frac{5x^3 - 2x^2 + 4x - 16}{2x^3 + 10x^2 - x + 4}$. What can we say about the end behavior of this function? Well, let's break this problem up into two smaller problems: What if we consider the numerator and the denominator separately? Find an end-behavior model for the numerator and a model for the denominator, and then take turns sharing your results with a partner.
  - *The function $5x^3$ is an end-behavior model for the numerator.*
  - *The function $2x^3$ is an end-behavior model for the denominator.*
- Now let's combine these observations: When we form the rational expression $f(x)$, does it seem reasonable that its end behavior is well approximated by the quotient $\frac{5x^3}{2x^3}$? And since $\frac{5x^3}{2x^3} = \frac{5}{2}$, it looks as though the outputs of f must get close to $\frac{5}{2}$ when x is very large. Use a calculator to verify this claim. Try, for example, using 10^{10} as an input to f.
  - $f(10^{10}) \approx 2.499\,999\,999$
- You have to admit that is pretty darn close to $\frac{5}{2}$.
- The general principle here is to focus on the term with the highest power, ignoring all of the other terms. Looking at the quotient of the two terms with the largest power gives you an idea of what the end behavior of a rational function is like.
- Let's take one last look at this example using factoring. Factor out the highest power of x from the numerator and denominator of $f(x) = \frac{5x^3 - 2x^2 + 4x - 16}{2x^3 + 10x^2 - x + 4}$. This should help you to see how all of the pieces in this discussion fit together.
  - *We have* $f(x) = \frac{5x^3 - 2x^2 + 4x - 16}{2x^3 + 10x^2 - x + 4} = \frac{x^3\left(5 - \frac{2}{x} + \frac{4}{x^2} - \frac{16}{x^3}\right)}{x^3\left(2 + \frac{10}{x} - \frac{1}{x^2} + \frac{4}{x^3}\right)}$.
  - *The quotient of $\frac{x^3}{x^3}$ is 1. The fractions in the expression are all approaching 0 as $x \to \infty$, so the expression $\frac{5 - \frac{2}{x} + \frac{4}{x^2} - \frac{16}{x^3}}{2 + \frac{10}{x} - \frac{1}{x^2} + \frac{4}{x^3}}$ is approaching $\frac{5}{2}$. This confirms our view that the function is approaching $\frac{5}{2}$ as $x \to \infty$.*

Example 2 (5 minutes)

- Now that we've seen that we can use our knowledge of polynomials to develop end-behavior models for rational functions, let's get some additional practice with this technique.
- Consider the function $f(x) = \frac{2x^3 + 3x - 1}{x^2 + x + 1}$. Find an end-behavior model for this function.
  - *The function $2x^3$ is an end-behavior model for the numerator.*
  - *The function x^2 is an end-behavior model for the denominator.*
  - *Thus, the expression $\frac{2x^3}{x^2} = 2x$ is an end-behavior model for f.*

- What conclusions can you draw about the end behavior of f based on your analysis?
  - *We know that as $x \to \infty$, $2x \to \infty$, so we conclude that $f(x) \to \infty$ as well.*
  - *Similarly, we know that as $x \to -\infty$, $2x \to -\infty$, so we conclude that $f(x) \to -\infty$ as well.*
- Use factoring to confirm that your analysis of the end behavior of f is correct.
  - $f(x) = \dfrac{2x^3 + 3x - 1}{x^2 + x + 1} = \dfrac{x^3 \cdot \left(2 + \frac{3}{x^2} - \frac{1}{x^3}\right)}{x^2 \cdot \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)} = x \cdot \dfrac{2 + \frac{3}{x^2} - \frac{1}{x^3}}{1 + \frac{1}{x} + \frac{1}{x^2}}$
  - *Each fraction is approaching 0 as $x \to \infty$, so the expression on the right is approaching $\frac{2}{1}$. This confirms that $y = 2x$ is an end-behavior model for $y = f(x)$.*
- So far, so good. Now let's use technology to further our understanding of this function. When you enter the function into your calculator, do the numerical results confirm your analysis?

x	100	500	$1{,}000$	$5{,}000$
$y = f(x)$	198	398	$1{,}998$	$9{,}998$

x	-100	-500	$-1{,}000$	$-5{,}000$
$y = f(x)$	-202	$-1{,}002$	$-2{,}002$	$-10{,}002$

  - *The numerical data confirms our views about the end behavior of f.*

Example 3 (4 minutes)

- Let's look at one final example together. What happens if we switch the numerator and the denominator in the previous example? We get the expression $f(x) = \dfrac{x^2 + x + 1}{2x^3 + 3x - 1}$. What do you suppose the end behavior of this function is like? Make a conjecture, and share it with a neighbor.
  - *In the original function, the outputs got very large. Since this function is the reciprocal of the previous example, perhaps the outputs are very small.*
- Find an end-behavior model for this function.
  - *The function x^2 is an end-behavior model for the numerator.*
  - *The function $2x^3$ is an end-behavior model for the denominator.*
  - *Thus, the expression $\dfrac{x^2}{2x^3} = \dfrac{1}{x}$ is an end-behavior model for f.*
- What conclusions can you draw about the end behavior of f based on your analysis?
  - *We know that as $x \to \infty$, $\dfrac{1}{x} \to 0$, so we conclude that $f(x) \to 0$ as well.*
  - *Similarly, we know that as $x \to -\infty$, $\dfrac{1}{x} \to 0$, so we conclude that $f(x) \to 0$ as well.*
- Use your calculator to confirm your results numerically.

x	-200	-100	100	200
$y = f(x)$	-0.0025	-0.0049	0.0050	0.0025

  - *The numerical data confirms our views about the end behavior of f.*

Exercises (5 minutes)

Give students time to work on the following exercises. Encourage students to compare their answers with a partner. Select three students to present their work to the class and to explain their thinking.

Exercises

Determine the end behavior of each rational function below.

1. $f(x) = \dfrac{7x^5 - 3x + 1}{4x^3 + 2}$

   This function has the same end behavior as

   $$\frac{7x^5}{4x^3} = \frac{7}{4}x^2.$$

   Using this model as a guide, we conclude that $f(x) \to \infty$ ***as*** $x \to \infty$ ***and that*** $f(x) \to \infty$ ***as*** $x \to -\infty$.

2. $f(x) = \dfrac{7x^3 - 3x + 1}{4x^3 + 2}$

   This function has the same end behavior as

   $$\frac{7x^3}{4x^3} = \frac{7}{4}.$$

   Using this model as a guide, we conclude that $f(x) \to \frac{7}{4}$ ***as*** $x \to \infty$ ***and that*** $f(x) \to \frac{7}{4}$ ***as*** $x \to -\infty$.

3. $f(x) = \dfrac{7x^3 + 2}{4x^5 - 3x + 1}$

   This function has the same end behavior as

   $$\frac{7x^3}{4x^5} = \frac{7}{4x^2}.$$

   Using this model as a guide, we conclude that $f(x) \to 0$ ***as*** $x \to \infty$ ***and that*** $f(x) \to 0$ ***as*** $x \to -\infty$.

Closing (1 minutes)

- Take a minute to write a summary in your notebook of what you learned today.
  - *We can use what we know about the end behavior of polynomials to analyze the end behavior of rational functions. The key point to understand is that the term with the highest exponent determines the end behavior of a polynomial.*
  - *We also explored how to describe the behavior of rational functions as they approach restricted input values.*

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 12: End Behavior of Rational Functions

Exit Ticket

Given $f(x) = \frac{x+2}{x^2-1}$, find the following, and justify your findings.

a. The end-behavior model for the numerator

b. The end-behavior model for the denominator

c. The end-behavior model for f

d. What is the value of $f(x)$ as $x \to \infty$?

e. What is the value of $f(x)$ as $x \to -\infty$?

f. What is the value of $f(x)$ as $x \to 1$ from the positive and negative sides?

g. What is the value of $f(x)$ as $x \to -1$ from the positive and negative sides?

Exit Ticket Sample Solutions

Given $f(x) = \frac{x+2}{x^2-1}$, find the following, and justify your findings.

a. **The end-behavior model for the numerator**

x

The end-behavior model is $f(x) = x^n$ where n is the greatest power of the expression. Here, the numerator is $x+2$, so $n = 1$, and the model is $f(x) = x$.

b. **The end-behavior model for the denominator**

x^2

The end-behavior model is $f(x) = x^n$ where n is the greatest power of the expression. Here, the denominator is $x^2 - 1$, so $n = 2$, and the model is $f(x) = x^2$.

c. **The end-behavior model for f**

$\frac{x}{x^2} = \frac{1}{x}$

I replaced the numerator and denominator with the end-behavior models of each and simplified.

d. **What is the value of $f(x)$ as $x \to \infty$?**

$f(x) \to 0$

As

$x \to \infty, f(x) \to \frac{1}{\infty} = 0.$

e. **What is the value of $f(x)$ as $x \to -\infty$?**

$f(x) \to 0$

As

$x \to -\infty, f(x) \to \frac{1}{-\infty} = 0.$

f. **What is the value of $f(x)$ as $x \to 1$ from the positive and negative sides?**

$f(x) \to \infty$ as $x \to 1$ from the positive side. As x nears 1 from the positive side, the numerator of the function approaches 3, and the denominator becomes a tiny positive number. The ratio of these is a very large positive number that exceeds any bounds.

$f(x) \to -\infty$ as $x \to 1$ from the negative side. As x nears 1 from the negative side, the numerator of the function approaches 3, and the denominator becomes a tiny negative number. The ratio of these is a very large negative number that exceeds any bounds.

g. **What is the value of $f(x)$ as $x \to -1$ from the positive and negative sides?**

$f(x) \to -\infty$ as $x \to -1$ from the positive side. As x nears -1 from the positive side, the numerator of the function approaches 1, and the denominator becomes a tiny negative number. The ratio of these is a very large negative number that exceeds any bounds.

$f(x) \to \infty$ as $x \to -1$ from the negative side. As x nears -1 from the negative side, the numerator of the function approaches 1, and the denominator becomes a tiny positive number. The ratio of these is a very large positive number that exceeds any bounds.

Problem Set Sample Solutions

1. **Analyze the end behavior of both functions.**

   a. $f(x) = x,\ g(x) = \frac{1}{x}$

   For $f(x)$: $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$.

   For $g(x)$: $g(x) \to \infty$ as $x \to 0$, and $g(x) \to -\infty$ as $x \to 0$.

   b. $f(x) = x^3,\ g(x) = \frac{1}{x^3}$

   For $f(x)$: $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$.

   For $g(x)$: $g(x) \to \infty$ as $x \to 0$, and $g(x) \to -\infty$ as $x \to 0$.

   c. $f(x) = x^2,\ g(x) = \frac{1}{x^2}$

   For $f(x)$: $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to \infty$ as $x \to -\infty$.

   For $g(x)$: $g(x) \to \infty$ as $x \to 0$, and $g(x) \to \infty$ as $x \to 0$.

   d. $f(x) = x^4,\ g(x) = \frac{1}{x^4}$

   For $f(x)$: $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$.

   For $g(x)$: $g(x) \to \infty$ as $x \to 0$, and $g(x) \to -\infty$ as $x \to 0$.

   e. $f(x) = x - 1,\ g(x) = \frac{1}{x-1}$

   For $f(x)$: $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$.

   For $g(x)$: $g(x) \to \infty$ as $x \to 0$, and $g(x) \to -\infty$ as $x \to 0$.

   f. $f(x) = x + 2,\ g(x) = \frac{1}{x+2}$

   For $f(x)$: $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$.

   For $g(x)$: $g(x) \to 0$ as $x \to \infty$, and $g(x) \to 0$ as $x \to -\infty$.

   g. $f(x) = x^2 - 4,\ g(x) = \frac{1}{x^2-4}$

   For $f(x)$: $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$.

   For $g(x)$: $g(x) \to \infty$ a $x \to 0$, and $g(x) \to -\infty$ as $x \to 0$.

2. **For the following functions, determine the end behavior. Confirm your answer with a table of values.**

a. $f(x) = \frac{3x-6}{x+2}$

f has the same end behavior as $\frac{3x}{x} = 3$. Using this model as a guide, we conclude that

$f(x) \to 3$ as $x \to \infty$, and $f(x) \to 3$ as $x \to -\infty$.

x	100	500	$1,000$	$5,000$
$y = f(x)$	2.882	2.976	2.988	2.998

x	-100	-500	$-1,000$	$-5,000$
$y = f(x)$	-3.122	3.024	3.012	3.002

b. $f(x) = \frac{5x+1}{x^2-x-6}$

f has the same end behavior as $\frac{5x}{x^2} = \frac{5}{x}$. Using this model as a guide, we conclude that

$f(x) \to 0$ as $x \to \infty$, and $f(x) \to 0$ as $x \to -\infty$.

x	100	500	$1,000$	$5,000$
$y = f(x)$	0.051	0.010	0.005	0.001

x	-100	-500	$-1,000$	$-5,000$
$y = f(x)$	-0.049	-0.010	-0.005	-0.001

c. $f(x) = \frac{x^3-8}{x^2-4}$

f has the same end behavior as $\frac{x^3}{x^2} = x$. Using this model as a guide, we conclude that

$f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$.

x	100	500	$1,000$	$5,000$
$y = f(x)$	100	500	$1,000$	$5,000$

x	-100	-500	$-1,000$	$-5,000$
$y = f(x)$	-100	-500	$-1,000$	$-5,000$

d. $f(x) = \frac{x^3-1}{x^4-1}$

f has the same end behavior as $\frac{x^3}{x^4} = \frac{1}{x}$. Using this model as a guide, we conclude that

$f(x) \to 0$ as $x \to \infty$, and $f(x) \to 0$ as $x \to -\infty$.

x	100	500	$1,000$	$5,000$
$y = f(x)$	0.010	0.002	0.001	0.0002

x	-100	-500	$-1,000$	$-5,000$
$y = f(x)$	-0.010	-0.002	-0.001	-0.0002

e. $f(x) = \frac{(2x+1)^3}{(x^2-x)^2}$

f has the same end behavior as $\frac{8x^3}{x^4} = \frac{8}{x}$. Using this model as a guide, we conclude that

$f(x) \to 0$ as $x \to \infty$, and $f(x) \to 0$ as $x \to -\infty$.

x	100	500	$1,000$	$5,000$
$y = f(x)$	0.083	0.016	0.008	0.0016

x	-100	-500	$-1,000$	$-5,000$
$y = f(x)$	-0.077	-0.016	-0.008	-0.0016

3. **For the following functions, determine the end behavior.**

a. $f(x) = \frac{5x^6 - 3x^3 + x - 2}{5x^4 - 3x^3 + x - 2}$

f has the same end behavior as $\frac{5x^6}{5x^4} = x^2$. Using this model as a guide, we conclude that

$f(x) \to \infty$ as $x \to \infty$, and $f(x) \to \infty$ as $x \to -\infty$.

b. $f(x) = \frac{5x^4 - 3x^3 + x - 2}{5x^6 - 3x^3 + x - 2}$

f has the same end behavior as $\frac{5x^4}{5x^6} = \frac{1}{x^2}$. Using this model as a guide, we conclude that

$f(x) \to 0$ as $x \to \infty$, and $f(x) \to 0$ as $x \to -\infty$.

c. $f(x) = \frac{5x^4 - 3x^3 + x - 2}{5x^4 - 3x^3 + x - 2}$

f has the same end behavior as $\frac{5x^4}{5x^4} = 1$. Using this model as a guide, we conclude that

$f(x) \to 1$ as $x \to \infty$, and $f(x) \to 1$ as $x \to -\infty$.

d. $f(x) = \frac{\sqrt{2}x^2 + x + 1}{3x + 1}$

f has the same end behavior as $\frac{\sqrt{2}x^2}{3x} = \frac{\sqrt{2}}{3}x$. Using this model as a guide, we conclude that

$f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$.

e. $f(x) = \frac{4x^2 - 3x - 7}{2x^3 + x - 2}$

f has the same end behavior as $\frac{4x^2}{2x^3} = \frac{2}{x}$.

$f(x) \to 0$ as $x \to \infty$, and $f(x) \to 0$ as $x \to -\infty$.

4. **Determine the end behavior of each function.**

   a. $f(x) = \frac{\sin(x)}{x}$

   $-1 \leq \sin x \leq 1$,

   $f(x) \to 0$ *as* $x \to \infty$, *and* $f(x) \to 0$ *as* $x \to -\infty$.

   b. $f(x) = \frac{\cos(x)}{x}$

   $-1 \leq \cos x \leq 1$,

   $f(x) \to 0$ *as* $x \to \infty$, *and* $f(x) \to 0$ *as* $x \to -\infty$.

   c. $f(x) = \frac{2^x}{x}$

   $f(x) \to \infty$ *as* $x \to \infty$, *and* $f(x) \to 0$ *as* $x \to -\infty$.

   d. $f(x) = \frac{x}{2^x}$

   $f(x) \to 0$ *as* $x \to \infty$, *and* $f(x) \to -\infty$ *as* $x \to -\infty$.

   e. $f(x) = \frac{4}{1+e^{-x}}$

   $f(x) \to 4$ *as* $x \to \infty$, *and* $f(x) \to 0$ *as* $x \to -\infty$.

   f. $f(x) = \frac{10}{1+e^{-x}}$

   $f(x) \to 10$ *as* $x \to \infty$, *and* $f(x) \to 0$ *as* $x \to -\infty$.

5. **Consider the functions $f(x) = x!$ and $g(x) = x^5$ for natural numbers x.**

   a. **What are the values of $f(x)$ and $g(x)$ for $x = 5, 10, 15, 20, 25$?**

   $f(x) = \{120, 3\,628\,800, 1\,307\,674\,368\,000, 2.4 \times 10^{18}, 1.6 \times 10^{25}\}$

   $g(x) = \{3125, 100\,000, 759\,375, 3\,200\,000, 9\,765\,625\}$

   b. **What is the end behavior of $f(x)$ as $x \to \infty$?**

   $f(x) \to \infty$

   c. **What is the end behavior of $g(x)$ as $x \to \infty$?**

   $g(x) \to \infty$

   d. **Make an argument for the end behavior of $\frac{f(x)}{g(x)}$ as $x \to \infty$.**

   Since $y = f(x)$ *increases so much faster than* $y = g(x)$, $f(x)$ *overpowers the division by* $g(x)$. *By* $x = 25$, $\frac{f(x)}{g(x)} \approx 1.6 \times 10^{18}$. *Thus,* $\frac{f(x)}{g(x)} \to \infty$.

e. Make an argument for the end behavior of $\frac{g(x)}{f(x)}$ as $x \to \infty$.

For the same reasons as in part (d), division by $f(x)$ overpowers $g(x)$ in the numerator. By $x = 25$, $\frac{g(x)}{f(x)} \approx 0.000\,000\,000\,000\,000$. Thus, $\frac{g(x)}{f(x)} \to 0$.

6. Determine the end behavior of the functions.

a. $f(x) = \frac{x}{x^2}$, $g(x) = \frac{1}{x}$

$f(x) \to 0$ as $x \to \infty$, and $f(x) \to 0$ as $x \to -\infty$.

$g(x) \to 0$ as $x \to \infty$, and $g(x) \to 0$ as $x \to -\infty$.

b. $f(x) = \frac{x+1}{x^2-1}$, $g(x) = \frac{1}{x-1}$

$f(x) \to 0$ as $x \to \infty$, and $f(x) \to 0$ as $x \to -\infty$.

$g(x) \to 0$ as $x \to \infty$, and $g(x) \to 0$ as $x \to -\infty$.

c. $f(x) = \frac{x-2}{x^2-x-2}$, $g(x) = \frac{1}{x+1}$

$f(x) \to 0$ as $x \to \infty$, and $f(x) \to 0$ as $x \to -\infty$.

$g(x) \to 0$ as $x \to \infty$, and $g(x) \to 0$ as $x \to -\infty$.

d. $f(x) = \frac{x^3-1}{x-1}$, $g(x) = x^2 + x + 1$

$f(x) \to \infty$ as $x \to \infty$, and $f(x) \to \infty$ as $x \to -\infty$.

$g(x) \to \infty$ as $x \to \infty$, and $g(x) \to \infty$ as $x \to -\infty$.

7. Use a graphing utility to graph the following functions f and g. Explain why they have the same graphs. Determine the end behavior of the functions and whether the graphs have any horizontal asymptotes.

a. $f(x) = \frac{x+1}{x-1}$, $g(x) = 1 + \frac{2}{x-1}$

The graphs are the same because the expressions are equivalent: $1 + \frac{2}{x-1} = \frac{x-1}{x-1} + \frac{2}{x-1} = \frac{x+1}{x-1}$ for all $x \neq 1$.

$f(x) \to 1$ as $x \to \infty$, and $f(x) \to 1$ as $x \to -\infty$.

$g(x) \to 1$ as $x \to \infty$, and $g(x) \to 1$ as $x \to -\infty$.

b. $f(x) = \frac{-2x+1}{x+1}$, $g(x) = \frac{3}{x+1} - 2$

The graphs are the same because the expressions are equivalent:
$\frac{3}{x+1} - 2 = \frac{3}{x+1} - \frac{2(x+1)}{x+1} = \frac{3-2x-2}{x+1} = \frac{-2x+1}{x+1}$ *for all $x \neq -1$*

$f(x) \to -2$ as $x \to \infty$, and $f(x) \to -2$ as $x \to -\infty$.

$g(x) \to -2$ as $x \to \infty$, and $g(x) \to -2$ as $x \to -\infty$.

Lesson 13: Horizontal and Vertical Asymptotes of Graphs of Rational Functions

Student Outcomes

- Students identify vertical and horizontal asymptotes of rational functions.

Lesson Notes

In this lesson, students continue to develop their understanding of the key features of rational functions. Students begin by connecting the algebraic and numeric work they did with end behavior in the previous lesson to the horizontal asymptote on the graph of a rational function. Students also analyze the behavior of a function as x approaches a value restricted from its domain. In this way, both horizontal and vertical asymptotes are defined. Students identify vertical and horizontal asymptotes without the use of technology and then use technology to confirm their results (**F-IF.C.7d**). While students have seen graphs of functions that contain a horizontal asymptote (i.e., exponential functions in Module 3 of Algebra I and Module 3 of Algebra II) and graphs of functions that contain vertical asymptotes (i.e., the tangent function in Module 2 of Algebra II and logarithmic functions in Module 3 of Algebra II), this is the first time that horizontal and vertical asymptotes are formally defined.

Classwork

Opening Exercise (5 minutes)

Allow students time to work the Opening Exercise independently. Then, have them compare answers with a partner before debriefing as a class.

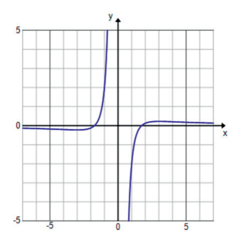
Opening Exercise

Determine the end behavior of each rational function below. Graph each function on the graphing calculator, and explain how the graph supports your analysis of the end behavior.

a. $f(x) = \frac{x^2 - 3}{x^3}$

End behavior: As $x \to \infty$, $f(x) \to 0$. As $x \to -\infty$, $f(x) \to 0$.

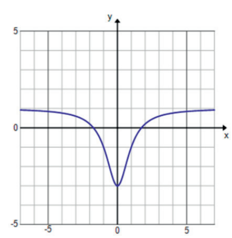
Looking at the ends of the graph, as x goes to ∞ or $-\infty$, y gets closer to 0.



b. $f(x) = \frac{x^2 - 3}{x^2 + 1}$

End behavior: As $x \to \infty$, $f(x) \to 1$. As $x \to -\infty$, $f(x) \to 1$.

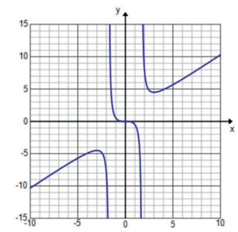
Looking at the ends of the graph, as x goes to ∞ or $-\infty$, y gets closer to 1.



c. $f(x) = \dfrac{x^3}{x^2 - 3}$

***End behavior:* As $x \to \infty$, $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to -\infty$.**

Looking at the ends of the graph, as x goes to ∞, y continues to increase toward ∞, and as x goes to $-\infty$, y continues to decrease toward $-\infty$.



Discussion (5 minutes)

- How is the end behavior of f related to the graph of f?
  - *The end behavior of f describes the value that y approaches as x approaches ∞ or $-\infty$.*
- When f approaches a particular number, L, as x approaches ∞ or $-\infty$, the line $y = L$ is called a *horizontal asymptote* on the graph of f.

Definition: Let L be a real number. The line given by $y = L$ is a horizontal asymptote of the graph of $y = f(x)$ if at least one of the following statements is true:

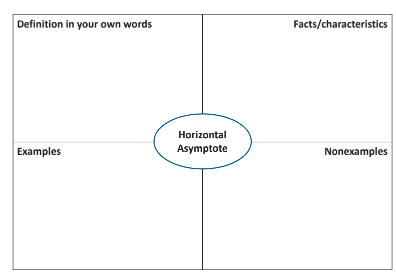
As $x \to \infty$, $f(x) \to L$.

As $x \to -\infty$, $f(x) \to L$.

- On a graph, an asymptote is sometimes drawn in as a dashed line. Draw the horizontal asymptotes for Exercises 1 and 2 on the graphs. Using this definition, identify the horizontal asymptote of each graph.
  - *For Exercise 1, the horizontal asymptote is $y = 0$. For Exercise 2, the horizontal asymptote is $y = 1$.*
- Can the graph of $y = f(x)$ actually cross through a horizontal asymptote?
  - *Looking at Exercise 1, the graph crosses the horizontal asymptote but then continues to approach 0 as x approaches ∞ or $-\infty$.*
- A graph may cross a horizontal asymptote once or many times, but its distance away from the horizontal asymptote must go to 0 as x approaches ∞ or $-\infty$.
- Look at the graph from Exercise 3. Why doesn't this graph have a horizontal asymptote? (Teacher note: These graphs are studied more closely in the next lesson. Students may note that the graph still seems to approach some boundary line, but that line is not horizontal.)
  - *Because as $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$. As x gets infinitely large, the y also gets infinitely large rather than approaching a particular value.*

Scaffolding:

If students are struggling, consider having them construct a Frayer model and then compare with a partner.

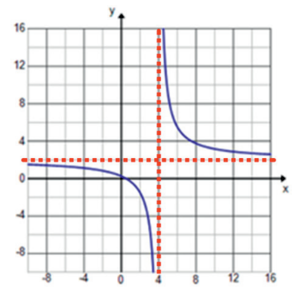


Example (5 minutes)

Give students time to work the example individually. Go over student responses, and use this as an opportunity to check for understanding. Then, have the Discussion that follows.

Example

Consider the rational function $f(x) = \dfrac{2x-1}{x-4}$.



a. **State the domain of f.**

D: set of all real numbers except $x = 4$

b. **Determine the end behavior of f.**

End behavior: As $x \to \infty, y \to 2$. As $x \to -\infty, y \to 2$.

c. **State the horizontal asymptote of the graph of $y = f(x)$.**

Horizontal asymptote: $y = 2$

d. **Graph the function on the graphing calculator, and make a sketch on your paper.**

Discussion (5 minutes)

- How is the domain of f related to the graph of $y = f(x)$?
  - *The graph did not cross through $x = 4$ because that value was removed from the domain of f. Since $f(4)$ is undefined, the graph of $y = f(x)$ cannot cross $x = 4$.*
- Describe the behavior of f as x approaches 4.
  - *As x approaches 4, the function approaches infinity on one side and negative infinity on the other side.*
- The line $x = 4$ is called a *vertical asymptote* of the graph of $y = f(x)$. Draw the vertical asymptote on your graph.
- In your own words, how would you define a vertical asymptote? (Let students articulate informal definitions either on paper or with a partner, and then write the following definition on the board.)
  - *A line representing a value of x that is restricted from the domain of f. A vertical line that a graph approaches but never crosses.*

MP.2

Definition: Let a be a real number. The line given by $x = a$ is a ***vertical asymptote*** of the graph of $y = f(x)$ if at least one of the following statements is true:

$$\text{As } x \to a,\ f(x) \to \infty.$$

$$\text{As } x \to a,\ f(x) \to -\infty.$$

- How could we identify that $x = 4$ is a vertical asymptote without using the graph?
  - *We could evaluate the function for values close to 4 to determine the behavior of f.*

MP.2

3.9	-68
3.99	-698
4	undefined
4.01	702
4.1	72

- *As x approaches 4 from numbers less than 4, y approaches $-\infty$. As x approaches 4 from numbers greater than 4, y approaches ∞.*

Exercises (17 minutes)

Allow students time to work in groups on the exercises, checking their work as they go using technology. Circulate the room, providing assistance as needed. Make sure that students are confirming the location of vertical asymptotes numerically as this will be a necessary skill when they graph rational functions without using technology. Allow time to share various responses to Exercises 7–9. Show the graphs to verify that the given functions have the correct characteristics.

Exercises

State the domain and end behavior of each rational function. Identify all horizontal and vertical asymptotes on the graph of each rational function. Then, verify your answer by graphing the function on the graphing calculator.

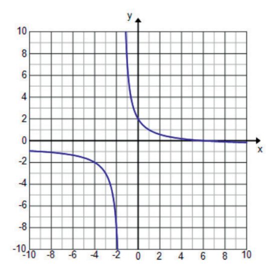
1. $f(x) = \dfrac{-x+6}{2x+3}$

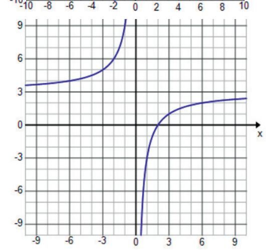
   ***D:* set of all real numbers except $x = -\frac{3}{2}$**

   ***Vertical asymptotes:* $x = -\frac{3}{2}$**

   ***End behavior:* As $\to \infty$, $y \to -\frac{1}{2}$. As $\to -\infty$, $y \to -\frac{1}{2}$.**

   ***Horizontal asymptote:* $y = -\frac{1}{2}$**





2. $f(x) = \dfrac{3x-6}{x}$

   ***D:* set of all real numbers except $x = 0$**

   ***Vertical asymptotes:* $x = 0$**

   ***End behavior:* As $x \to \infty$, $y \to 3$. As $x \to -\infty$, $y \to 3$.**

   ***Horizontal asymptote:* $y = 3$**

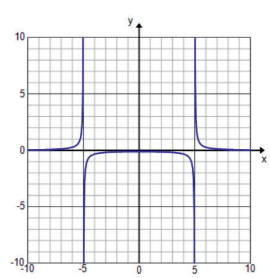
3. $f(x) = \dfrac{3}{x^2-25}$

   ***D:* set of all real numbers except $x = -5, 5$**

   ***Vertical asymptotes:* $x = -5$ and $x = 5$**

   ***End behavior:* As $x \to \infty$, $y \to 0$. As $x \to -\infty$, $y \to 0$.**

   ***Horizontal asymptote:* $y = 0$**



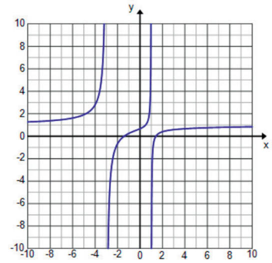
4. $f(x) = \dfrac{x^2 - 2}{x^2 + 2x - 3}$

D: set of all real numbers except $x = -3, 1$

Vertical asymptotes: $x = -3$ and $x = 1$

End behavior: As $x \to \infty$, $y \to 1$. As $x \to -\infty$, $y \to 1$.

Horizontal asymptote: $y = 1$



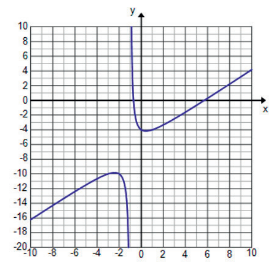
5. $f(x) = \dfrac{x^2 - 5x - 4}{x + 1}$

D: set of all real numbers except $x = -1$

Vertical asymptote: $x = -1$

End behavior: As $x \to \infty$, $y \to \infty$. As $x \to -\infty$, $y \to -\infty$.

Horizontal asymptote: none



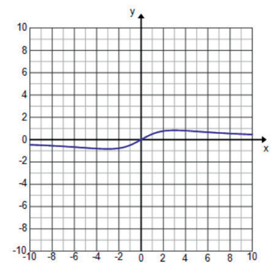
6. $f(x) = \dfrac{5x}{x^2 + 9}$

D: set of all real numbers

Vertical asymptote: none

End behavior: As $x \to \infty$, $y \to 0$. As $x \to -\infty$, $y \to 0$.

Horizontal asymptote: $y = 0$



Write an equation for a rational function whose graph has the given characteristic. Graph your function on the graphing calculator to verify.

7. **A horizontal asymptote of $y = 2$ and a vertical asymptote of $x = -2$**

$f(x) = \dfrac{2x + 5}{x + 2}$

8. **A vertical asymptote of $x = 6$ and no horizontal asymptote**

$f(x) = \dfrac{x^2}{x - 6}$

9. **A horizontal asymptote of $y = 6$ and no vertical asymptote**

$f(x) = \dfrac{6x^2}{x^2 + 1}$

MP.7

- When looking at a rational function, what information does the structure of the function give you about the horizontal asymptote of its graph?
  - *When the numerator has a higher exponent, the graph does not have a horizontal asymptote. When the denominator has a higher exponent, the graph has a horizontal asymptote of $y = 0$. When the highest exponent in the numerator and the denominator is the same, the graph has a horizontal asymptote $y = L$ where L is not zero.*

MP.7

- What information does the structure of the function give you about the vertical asymptote of its graph?
  - *If the denominator does not equal zero for any real number x, then the graph does not have a vertical asymptote.*
- There are many possible answers for Exercise 7; what did every function need to have in common?
  - *There needed to be a factor of $x + 2$ in the denominator. The leading coefficient in the numerator needed to be 2.*

Closing (3 minutes)

Use the following questions to review the key points from the lesson.

- For a value outside of the domain of a rational function, what could potentially happen on the graph of the function?
  - *There could be a vertical asymptote.*
- If there is a vertical asymptote at $x = a$, then as x approaches a, what must $f(x)$ approach?
  - *Either infinity or negative infinity*
- How can we tell whether $f(x)$ approaches infinity or negative infinity?
  - *By filling in a test value on either side of the vertical asymptote to see if the output value is a large negative number or a large positive number*
- How can we determine if the graph of a rational function has a horizontal asymptote?
  - *By examining its end behavior: If the function approaches a particular number L as x approaches infinity or negative infintity, then the line $y = L$ is a horizontal asymptote on the graph.*

Lesson Summary

- **Let a be a real number. The line given by $x = a$ is a *vertical asymptote* of the graph of $y = f(x)$ if at least one of the following statements is true:**
  - **As $x \to a$, $f(x) \to \infty$.**
  - **As $x \to a$, $f(x) \to -\infty$.**
- **Let L be a real number. The line given by $y = L$ is a *horizontal asymptote* of the graph of $y = f(x)$ if at least one of the following statements is true:**
  - **As $x \to \infty$, $f(x) \to L$.**
  - **As $x \to -\infty$, $f(x) \to L$.**

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 13: Horizontal and Vertical Asymptotes of Graphs of Rational Functions

Exit Ticket

Consider the function $f(x) = \frac{-2x+5}{x^2-5x-6}$.

1. Looking at the structure of the function, what information can you gather about the graph of f?

2. State the domain of f.

3. Determine the end behavior of f.

4. State the equations of any vertical and horizontal asymptotes on the graph of $y = f(x)$.

Exit Ticket Sample Solutions

Consider the function $f(x) = \frac{-2x+5}{x^2-5x-6}$.

1. Looking at the structure of the function, what information can you gather about the graph of f?

   The graph has a horizontal asymptote at $y = 0$ because the denominator has a higher exponent than the numerator. There are two values of x that cause the denominator to equal zero, so the graph potentially has two vertical asymptotes.

2. State the domain of f.

   D: the set of all real numbers except $x = 6, -1$

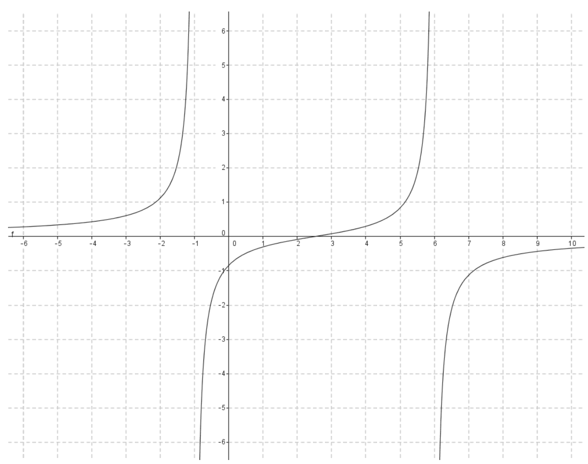
3. Determine the end behavior of f.

   End behavior: As $x \to \infty$, $y \to 0$. As $x \to -\infty$, $y \to 0$.

4. State the equations of any vertical and horizontal asymptotes on the graph of $y = f(x)$.

   Vertical asymptotes: $x = 6$ and $x = -1$

   Horizontal asymptote: $y = 0$



Problem Set Sample Solutions

1. State the domain of each rational function. Identify all horizontal and vertical asymptotes on the graph of each rational function.

   a. $y = \frac{3}{x^3 - 1}$

   The domain is all real numbers except $x = 1$, which is a vertical asymptote. The horizontal asymptote is $y = 0$.

b. $y = \dfrac{2x+2}{x-1}$

The domain is all real numbers except $x = 1$, which is a vertical asymptote. The graph of the function has a horizontal asymptote at $y = 2$.

c. $y = \dfrac{5x^2 - 7x + 12}{x^3}$

The domain is all real numbers except $x = 0$, which is a vertical asymptote. The horizontal asymptote occurs at $y = 0$.

d. $y = \dfrac{3x^6 - 2x^3 + 1}{16 - 9x^6}$

The domain is all real numbers except $x = \pm\sqrt[3]{\frac{4}{3}}$, which are both vertical asymptotes. The horizontal asymptote is $y = -\frac{1}{3}$.

e. $f(x) = \dfrac{6-4x}{x+5}$

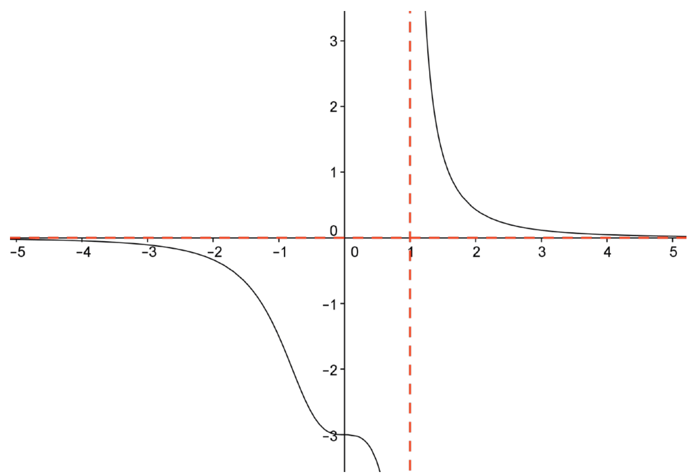
The domain is all real numbers except $x = -5$, which is a vertical asymptote. The horizontal asymptote is the line $y = -4$.

f. $f(x) = \dfrac{4}{x^2 - 4}$

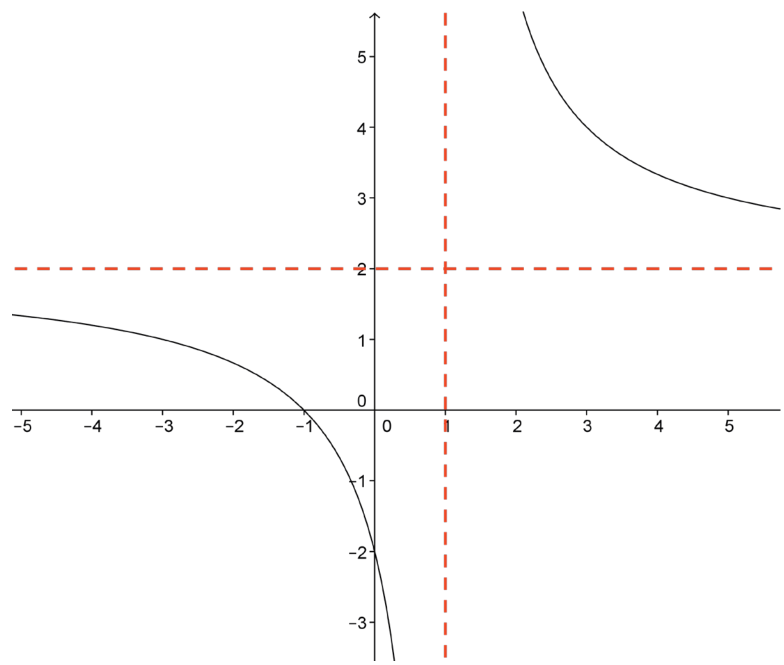
The domain is all real numbers except $x = -2$ and $x = 2$, which are both vertical asymptotes. The horizontal asymptote is the line $y = 0$.

2. **Sketch the graph of each function in Exercise 1 with asymptotes and excluded values from the domain drawn on the graph.**

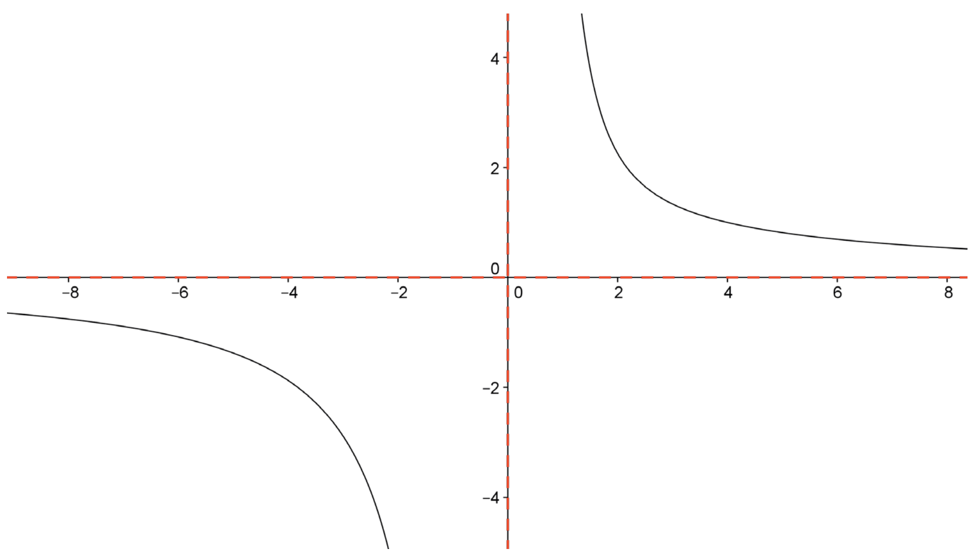
a.



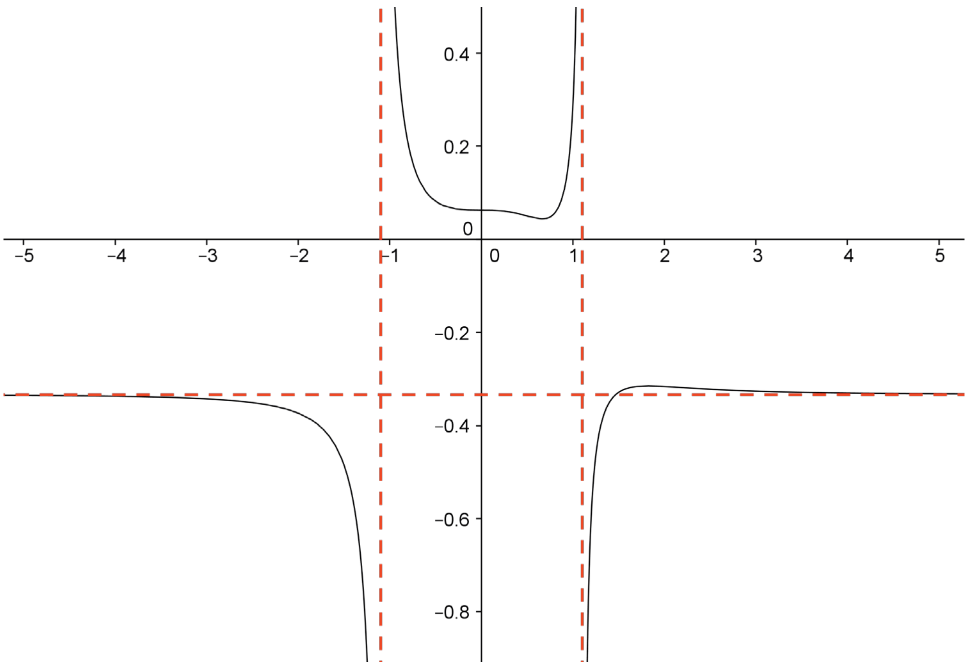
b.



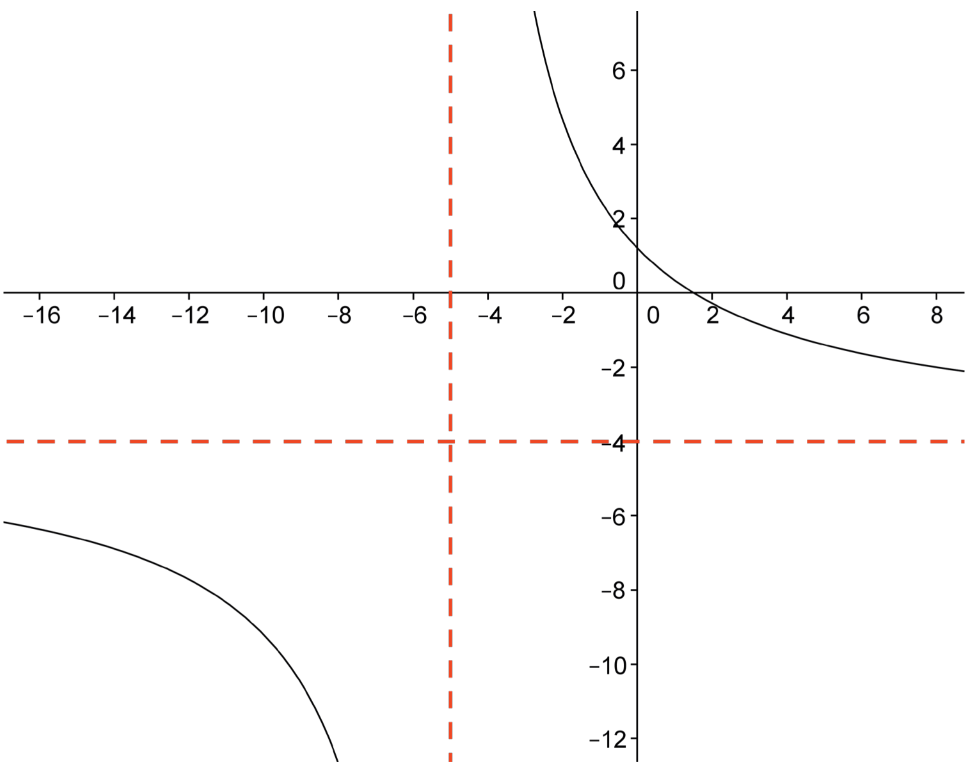
c.



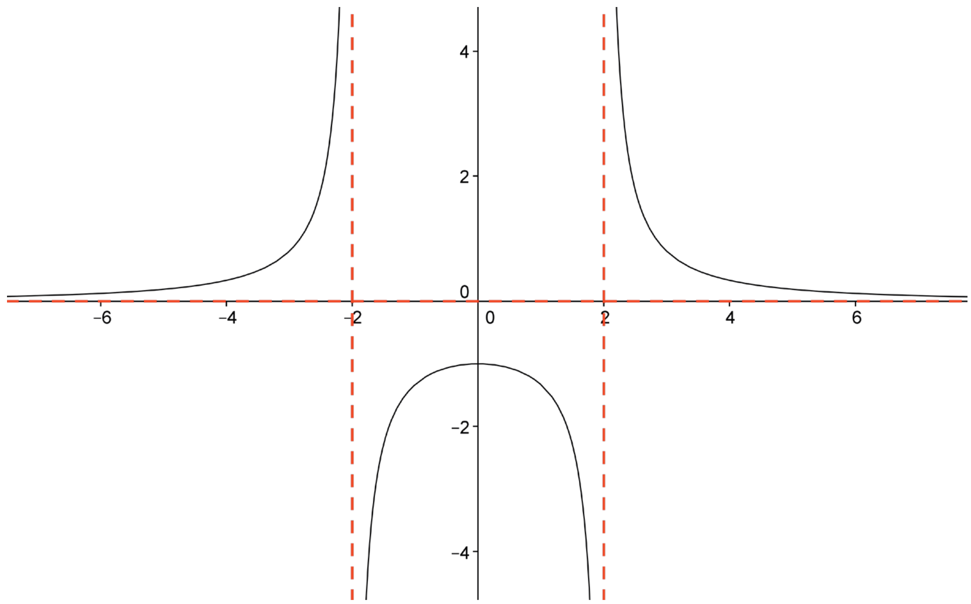
d.


0.4
0.2
0
−5
−4
−3
−2
−1
0
1
2
3
4
5
−0.2
−0.4
−0.6
−0.8

e.


6
4
2
0
−16
−14
−12
−10
−8
−6
−4
−2
0
2
4
6
8
−2
−4
−6
−8
−10
−12

f.



3. Factor out the highest power of x in each of the following, and cancel common factors if you can. Assume x is nonzero.

a. $y = \dfrac{x^3 + 3x - 4}{3x^3 - 4x^2 + 2x - 5}$

$$y = \frac{x^3\left(1 + \frac{3}{x^2} - \frac{4}{x^3}\right)}{x^3\left(3 - \frac{4}{x} + \frac{2}{x^2} - \frac{5}{x^3}\right)}$$

$$= \frac{1 + \frac{3}{x^2} - \frac{4}{x^3}}{3 - \frac{4}{x} + \frac{2}{x^2} - \frac{5}{x^3}}$$

b. $y = \dfrac{x^3 - x^2 - 6x}{x^3 + 5x^2 + 6x}$

$$y = \frac{x^3\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)}{x^3\left(1 + \frac{5}{x} + \frac{6}{x^2}\right)}$$

$$= \frac{1 - \frac{1}{x} - \frac{6}{x^2}}{1 + \frac{5}{x} + \frac{6}{x^2}}$$

c. $y = \dfrac{2x^4 - 3x + 1}{5x^3 - 8x - 1}$

$$y = \frac{x^4\left(2 - \frac{3}{x^3} + \frac{1}{x^4}\right)}{x^3\left(5 - \frac{8}{x^2} - \frac{1}{x^3}\right)}$$
$$= \frac{x\left(2 - \frac{3}{x^3} + \frac{1}{x^4}\right)}{5 - \frac{8}{x^2} - \frac{1}{x^3}}$$

d. $y = -\dfrac{9x^5 - 8x^4 + 3x + 72}{7x^5 + 8x^4 + 8x^3 + 9x^2 + 10x}$

$$y = -\frac{x^5\left(9 - \frac{8}{x} + \frac{3}{x^4} + \frac{72}{x^5}\right)}{x^5\left(7 + \frac{8}{x} + \frac{8}{x^2} + \frac{9}{x^3} + \frac{10}{x^4}\right)}$$
$$= -\frac{9 - \frac{8}{x} + \frac{3}{x^4} + \frac{72}{x^5}}{7 + \frac{8}{x} + \frac{8}{x^2} + \frac{9}{x^3} + \frac{10}{x^4}}$$

e. $y = \dfrac{3x}{4x^2+1}$

$$y = \frac{x(3)}{x^2\left(4 + \frac{1}{x^2}\right)}$$
$$= \frac{3}{x\left(4 + \frac{1}{x^2}\right)}$$

4. **Describe the end behavior of each function in Exercise 3.**

a. *As* $x \to \pm\infty$, $y \to \frac{1}{3}$.

b. *As* $x \to \pm\infty$, $y \to 1$.

c. *As* $x \to \pm\infty$, $y \to \pm\infty$.

d. *As* $x \to \pm\infty$, $y \to -\frac{9}{7}$.

e. *As* $x \to \pm\infty$, $y \to 0$.

5. **Using the equations that you wrote in Exercise 3, make some generalizations about how to quickly determine the end behavior of a rational function.**

The coefficients on the leading terms always end up being the horizontal asymptotes when the functions have the same power. When the power in the numerator is larger, then the functions either increase or decrease without bound as x increases or decreases without bound. When the power in the denominator is larger, then the functions go to zero.

6. **Describe how you may be able to use the end behavior of the graphs of rational functions, along with the excluded values from the domain and the equations of any asymptotes, to graph a rational function without technology.**

Let f be a rational function. The end behavior of the graph of $y = f(x)$ tells you in what direction the graph of f is heading, and the process of finding the end behavior yields a simpler function that f gets close to as x increases in magnitude. The vertical asymptotes and excluded values from the domain tell you where the graph varies close to the origin. All of this information combined gives you a very good idea of what the graph of f looked like without technology.

Lesson 14: Graphing Rational Functions

Student Outcomes

- Students graph rational functions showing intercepts, asymptotes, and end behavior.

Lesson Notes

In this lesson, students continue to explore the key features of the graphs of rational functions and use the intercepts and asymptotes to create graphs of rational functions without using technology (**F-IF.C.7d**). They also realize that not all values excluded from the domain of x result in a vertical asymptote. Emphasize to students that techniques learned in calculus would be needed to produce the finer details of the graph (like the relative maximum point or the changes in the curvature of the graph). However, a general idea of the graph of a rational function can still be produced by using its key features. Students graph some rational functions whose boundary lines are not horizontal, but they are not required to find equations for these asymptotes. Point out that while the graphs are not approaching a particular y-value, the graph does approach some boundary line (or curve). However, this is not the focal point of the lesson.

Classwork

Opening Exercise (4 minutes)

Allow students time to complete the Opening Exercise individually, and then discuss results as a class. Consider allowing students to check their answers using technology.

Opening Exercise

State the domain of each of the following functions. Then, determine whether or not the excluded value(s) of x are vertical asymptotes on the graph of the function. Give a reason for your answer.

a. $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$

D: the set of all real numbers except $x = 2$

$x = 2$ is not a vertical asymptote on the graph of f because as $x \to 2$, $f(x) \to 1$.

b. $f(x) = \dfrac{x^2 + 3x + 2}{x - 2}$

D: the set of all real numbers except $x = 2$

$x = 2$ is a vertical asymptote on the graph of f because as $x \to 2$, $f(x) \to \infty$ on one side and $-\infty$ on one side.

- When a value of x is excluded from the domain, is that value always going to correspond to a vertical asymptote on the graph?
  - *No. In example 1, $x = 2$ is excluded from the domain, but the graph did not have a vertical asymptote at $x = 2$.*

- How did you decide whether or not the graph had a vertical asymptote at $x = 2$?
  - *I made a table and filled in values of x that were close to 2.*

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	0.9	0.99	0.999	Undefined	1.001	1.01	1.1

  - *As $x \to 2$, $f(x) \to 1$. Therefore, $x = 2$ cannot be a vertical asymptote.*

If necessary, remind students about the definition of a vertical asymptote. Since $f(x)$ does not approach ∞ or $-\infty$ as x approaches 2, there cannot be a vertical asymptote at $x = 2$.

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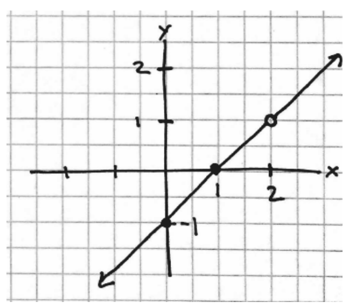
- Can this function be rewritten as a simpler function?
  - *Yes.* $f(x) = \frac{x^2 - 3x + 2}{x - 2} = \frac{(x-2)(x-1)}{x-2} = x - 1, x \neq 2$
- So, for every value of x, except 2, $f(x) = x - 1$. What does this graph look like?
  - *The graph is a line with a y-intercept of 1 and an x-intercept of 1.*
- What is the only difference between the graph of $f(x) = \frac{x^2 - 3x + 2}{x - 2}$ and the graph of $y = x - 1$?
  - *At $x = 2$, the point $(2,1)$ is missing from the graph of f.*
- These functions are sometimes called two functions that agree at all but one point. (Have students graph the function from Exercise 1 either on paper or using the graphing calculator.)



- What was different about Exercise 2?
  - *The domain was the same, but this time, there was a vertical asymptote at $x = 2$.*
- How did you know there was a vertical asymptote?
  - *I made a table and filled in values of x that were close to 2.*

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	-113.1	$-1{,}193$	$-1{,}1993$	Undefined	12,007	1,207	127.1

  - As $x \to 2$, $f(x) \to \infty$ and $-\infty$; therefore, there is a vertical asymptote at $x = 2$.
- Can this function be reduced to a simpler function?

  No. That is why there is a vertical asymptote rather than a point missing from the graph.

Example 1 (7 minutes)

Work through the example with the class, giving students time to identify each key feature. Construct the graph as students find each key feature. After students have found the key features, work with the class to complete the graph. Often, students can find the key features but have difficulty producing the graph. Do not allow students to use the graphing calculator to sketch the graph.

> *Scaffolding:*
> If students are struggling, consider starting with an easier example such as the following:
> $$f(x) = \frac{3x-6}{x-2}$$

- Where should we start?
  - *Identify the domain.*
- What do we now know about the graph of f?
  - *There are vertical asymptotes at $x = -4$ and $x = 4$.*
- How do we know that these are vertical asymptotes and not points missing from the graph?
  - *The function could not be reduced to a simpler function. We can also confirm by determining if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to -4$ or 4.*
- How can we determine if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to -4$ or 4? What number should we fill in?
  - *We need to evaluate the function at numbers close to 4 and -4. It does not really matter, but it should be a number close to -4 or 4.*

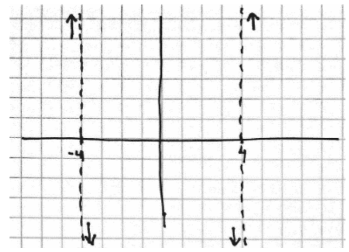
Point out to students that we do not really care about the actual value. We are looking to see if we get a large positive output or a large negative output.

- What else do we need to know?
  - *The end behavior and the horizontal asymptote (if the graph of f has one)*
- We need some points that lie on the graph. Where should we start?
  - *x-intercept(s) and y-intercept*
- Let's examine what we have at this point. Is this enough information to sketch the graph?
  - *We have the broad features of the graph, so we have enough information to make a good guess at the shape of the graph. We could always fill in more test values to find additional points on the graph if we wanted to increase its accuracy.*

Example 1

Sketch the graph of the rational function $f(x) = \frac{2x^2 - x}{x^2 - 16}$ showing all the key features of the graph. Label the key features on your graph.

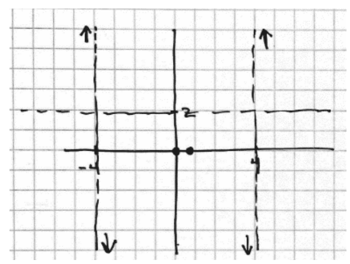
D: set of all real numbers except $x = 4, -4$



Vertical asymptotes: $x = 4$ and $x = -4$

Test values near the vertical asymptotes:

x	$f(x)$
-4.1	46.5679
-4	*undefined*
-3.9	-43.443
3.9	-33.5696
4	*undefined*
4.1	36.4444

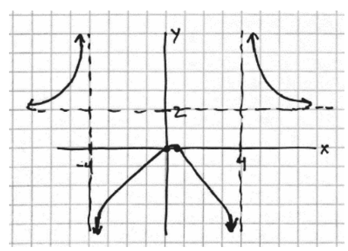


End behavior: As $x \to \infty$, $y \to 2$.

Horizontal asymptote: $y = 2$

x-intercepts: $2x^2 - x = 0 \to x = 0, \frac{1}{2} \to (0, 0)$ and $\left(\frac{1}{2}, 0\right)$

y-intercept: $(0, 0)$



Example 2 (7 minutes)

Give students time to analyze the function either individually or in pairs before discussing the graph as a class.

- After analyzing the behavior on either side of the vertical asymptote and plotting the intercepts, do we know the exact shape of the graph?
  - *No. We do not know which way to make the graph curve or if there are "bumps" or turning points.*
- How can we use the end behavior to complete the graph?
  - *Using the fact that as $x \to \infty$, $y \to \infty$, I know the arrow on the right must be pointing up. And $x \to -\infty$, $y \to -\infty$, tells me the arrow on the left must point down.*

Scaffolding:

- For students who are struggling, consider starting with an easier example such as $f(x) = \frac{x^2-1}{x}$.
- As an extension, students who like a challenge could be asked to find the equation of that boundary line first by estimating and then by dividing to rewrite the function.

$$f(x) = \frac{x^2 + 5x - 6}{x + 1} = x + 4 - \frac{10}{x + 1}$$

As $x \to \infty$, $\frac{10}{x+1} \to 0$, so $f(x) \to x + 4$.

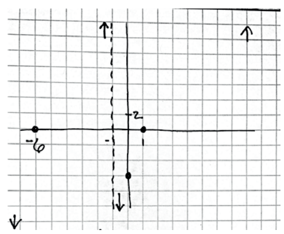
The function approaches the line $y = x + 4$.

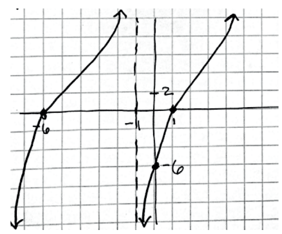
Have students draw a graph. Share a few samples of student work. Then, either display the graph using a graphing utility, or have students graph the function using a graphing calculator to compare with their graphs. Students may see that their graph was not quite right but displayed the correct key features (as shown below).

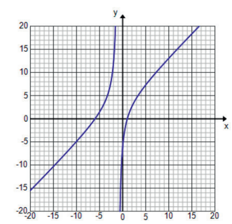
- Does the graph appear to have some boundary that it is approaching as $x \to \infty$ or $-\infty$?
  - *Yes. Even though as x approaches infinity, $f(x)$ also approaches infinity, there appears to be some line that the graph of f is approaching as x gets infinitely large.*

Example 2

Graph the function $f(x) = \dfrac{x^2 + 5x - 6}{x + 1}$ showing all the key features.







***D:* set of all real numbers except $x = -1$**

***Vertical asymptote:* $x = -1$**

Test values near the vertical asymptote:

x	$f(x)$
-1.1	102.9
-1	*undefined*
-0.9	-96.9

***End behavior:* As $x \to \infty$, $y \to \infty$. As $x \to -\infty$, $y \to -\infty$.**

***x-intercepts:* $x^2 + 5x - 6 = 0 \to x = -6, 1 \to (-6, 0)$ and $(1, 0)$**

***y-intercept:* $(0, -6)$**

Exercises (20 minutes)

Allow students time to work in groups on the exercises using technology to check their work as they go. If graphing calculators are not being used, display the graphs as students complete each exercise. Circulate the room, providing assistance as needed. If time is short, students need not complete all eight exercises, but it is valuable to see a variety of graphs in order to compare. Assign different groups different exercise numbers, ensuring that each group is seeing a variety of graphs. When debriefing, emphasize the structure of the functions and what information that provides about its graph. Compare and contrast graphs of functions whose equations looked similar. Use the questions that follow the exercises as a guide.

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Exercises

Sketch the graph of each rational function showing all the key features. Verify your graph by graphing the function on the graphing calculator.

1. $f(x) = \frac{4x-6}{2x+5}$

D: set of all real numbers except $x = -\frac{5}{2}$

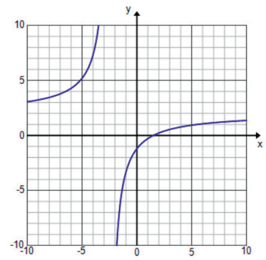
Vertical asymptotes: $x = -\frac{5}{2}$

End behavior: As $x \to \infty$, $y \to 2$.

Horizontal asymptote: $y = 2$

x-intercept: $\left(\frac{3}{2}, 0\right)$

y-intercept: $\left(0, -\frac{6}{5}\right)$



2. $f(x) = \frac{(3x-6)(x-4)}{x(x-4)}$

Equivalent form: $f(x) = \frac{3x-6}{x}$ *with* $x \neq 4$

D: set of all real numbers except $x = 0, 4$

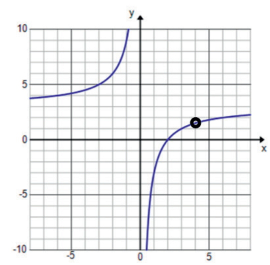
Vertical asymptotes: $x = 0$

End behavior: As $x \to \infty$, $y \to 3$.

Horizontal asymptote: $y = 3$

x-intercept: $(2, 0)$

y-intercept: none



3. $f(x) = \frac{3x-2x^2}{x-2}$

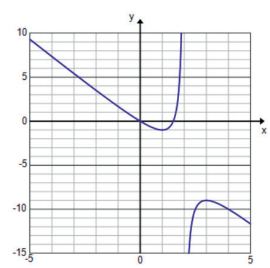
D: set of all real numbers except $x = 2$

Vertical asymptotes: $x = 2$

End behavior: As $x \to \infty$, $y \to -\infty$. *As* $x \to -\infty$, $y \to \infty$.

x-intercept: $(0, 0)$ *and* $\left(\frac{3}{2}, 0\right)$

y-intercept: $(0, 0)$



4. $f(x) = \frac{x-2}{3x-2x^2}$

D: set of all real numbers except $x = 0, \frac{3}{2}$

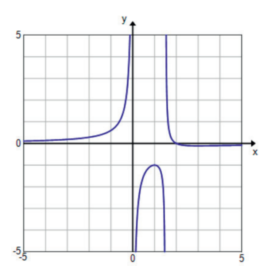
Vertical asymptotes: $x = 0$ *and* $x = \frac{3}{2}$

End behavior: As $x \to \infty$, $y \to 0$.

Horizontal asymptote: $y = 0$

x-intercept: $(2, 0)$

y-intercept: none



5. $f(x) = \dfrac{x}{x^2 - 9}$

D: set of all real numbers except $x = -3, 3$

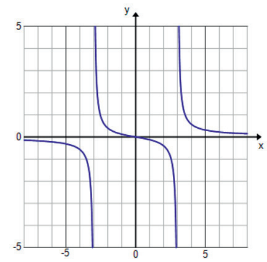
Vertical asymptotes: $x = -3$ *and* $x = 3$

End behavior: As $x \to \infty$, $y \to 0$.

Horizontal asymptote: $y = 0$

x-intercept: $(0, 0)$

y-intercept: $(0, 0)$



6. $f(x) = \dfrac{x^2}{x^2 - 9}$

D: set of all real numbers except $x = -3, 3$

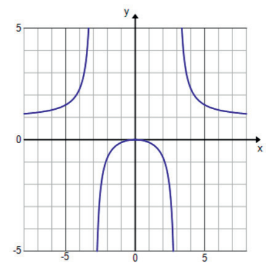
Vertical asymptotes: $x = -3$ *and* $x = 3$

End behavior: As $x \to \infty$, $y \to 1$.

Horizontal asymptote: $y = 1$

x-intercept: $(0, 0)$

y-intercept: $(0, 0)$



7. $f(x) = \dfrac{x^2 - 9}{x}$

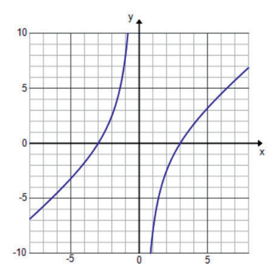
D: set of all real numbers except $x = 0$

Vertical asymptotes: $x = 0$

End behavior: As $x \to \infty$, $y \to \infty$. *As* $x \to -\infty$, $y \to -\infty$.

x-intercept: $(-3, 0)$ *and* $(3, 0)$

y-intercept: none



8. $f(x) = \dfrac{x^2 - 9x}{x}$

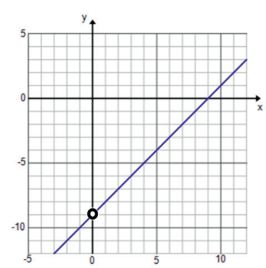
D: set of all real numbers except $x = 0$

Equivalent function: $f(x) = x - 9$, $x \neq 0$

End behavior: As $x \to \infty$, $y \to \infty$. *As* $x \to -\infty$, $y \to -\infty$.

x-intercept: $(9, 0)$

y-intercept: none



9. $f(x) = \dfrac{x^3 - 8}{x - 2}$

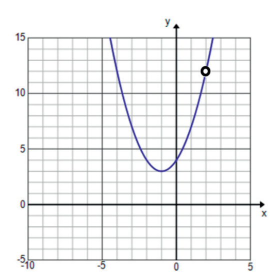
D: set of all real numbers except $x = 2$

Equivalent function: $f(x) = x^2 + 2x + 4$, $x \neq 2$

End behavior: As $x \to \infty$, $y \to \infty$. *As* $x \to -\infty$, $y \to \infty$.

x-intercept: none

y-intercept: $(0, 4)$



10. $f(x) = \frac{x^3 - 8}{x - 1}$

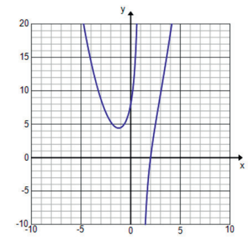
D: set of all real numbers except $x = 1$

Vertical asymptote: $x = 1$

End behavior: As $x \to \infty$, $y \to \infty$. As $x \to -\infty$, $y \to -\infty$.

x-intercept: $(2, 0)$

y-intercept: $(0, 8)$



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- For Exercises 1 and 2, how did switching the numerator and denominator change the graph?
  - *The x-intercepts from Exercise 1 became the vertical asymptotes for the graph in Exercise 2 and vice versa. The end behavior changed. In Exercise 1, the function increased and decreased without bound as x approached infinity. In Exercise 2, the function approached 0 as x approached infinity.*
- Looking at Exercises 3 and 4, the only difference is that the numerator changes from x to x^2. Compare and contrast these two graphs.
  - *The two graphs have the same vertical asymptotes and the same intercepts. The horizontal asymptotes are different, and the behavior of the function changes as x approaches -3 and 3.*
- For Exercises 5 and 6, the two functions have the same domain. What is different about the two graphs?
  - *In Exercise 6, the function could be simplified. Instead of having a vertical asymptote at $x = 0$ like in Exercise 5, the graph has a single point missing.*
- Looking at the graphs from Exercises 7 and 8, did changing the denominator from $x - 2$ to $x - 1$ significantly change the graph?
  - *Yes. In Exercise 7, the function could be reduced to a quadratic expression, so the graph was a parabola with a single point missing at $x = 2$. For Exercise 8, the function could not be reduced, and the graph had a vertical asymptote at $x = 1$. Also, the intercepts changed, and the end behavior was different as x approached negative infinity.*

Closing (2 minutes)

Have students summarize the key features of the graph of a rational function and the steps taken to graph a rational function either in writing or with a partner. Then, share responses as a class.

- What are the key features of the graph of a rational function?
  - *End behavior, vertical and horizontal asymptotes, and x- and y-intercepts*
- How do we use the function to locate these key features?
  - *See if the function can be reduced to a simpler function by factoring if possible.*
  - *Identify the vertical asymptote(s) by determining where the denominator is equal to zero.*
  - *Use test values on either side of the vertical asymptote to determine the direction of the graph.*
  - *Analyze the end behavior by analyzing what happens to f as x gets very large or very small.*
  - *Identity the horizontal asymptote (if applicable) by looking at the degree of the numerator and denominator and determining what happens as x gets very large or very small.*
  - *Find the x-intercept(s) by setting $f(x) = 0$. Find the y-intercept by evaluating $f(0)$.*

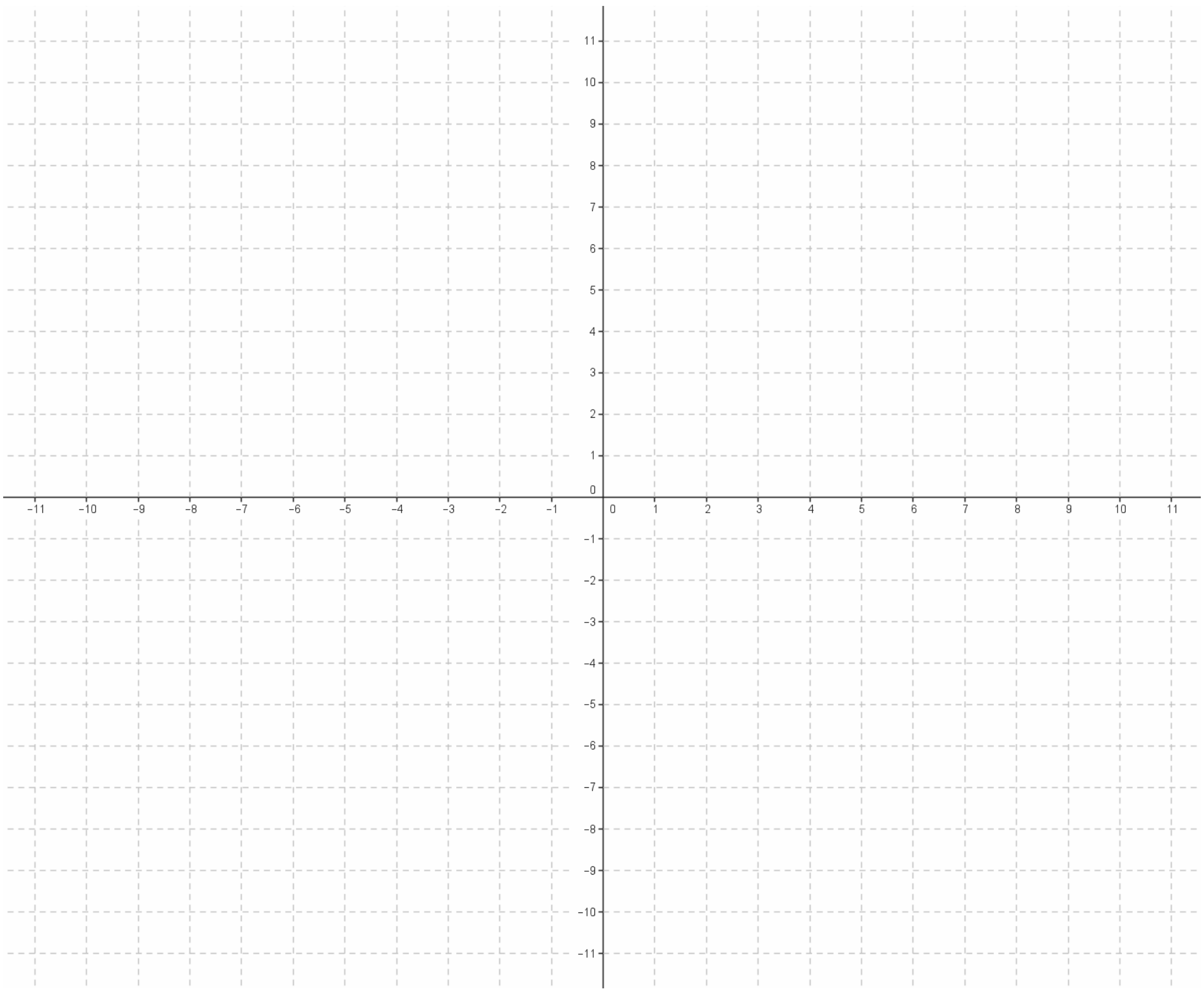
Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 14: Graphing Rational Functions

Exit Ticket

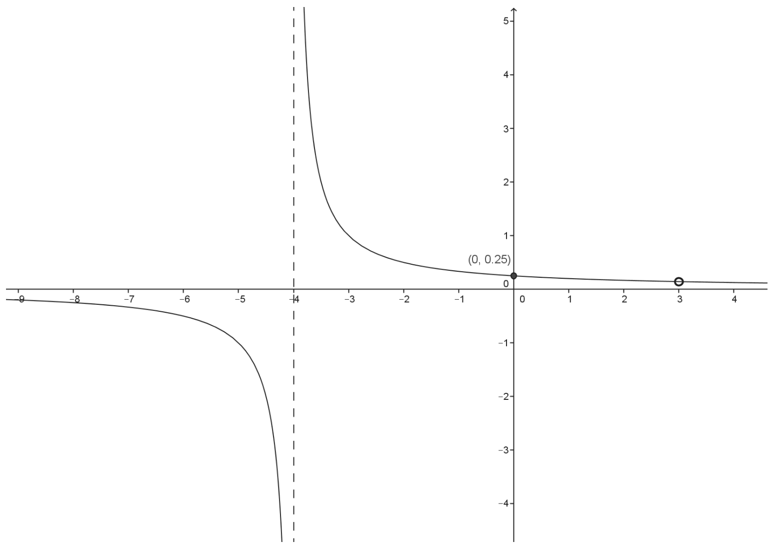
1. Sketch the graph of $y = \dfrac{x-3}{x^2+x-12}$. Label horizontal and vertical asymptotes, and identify any discontinuities, x-intercepts, and the y-intercept if they exist. Describe the end behavior of the function.



2. Does the graph of the function $f(x) = \dfrac{x^2 - 8x - 9}{x + 1}$ have a vertical asymptote or a point missing at $x = -1$? Explain your reasoning, and support your answer numerically.

Exit Ticket Sample Solutions

1. **Sketch the graph of $y = \frac{x-3}{x^2 + x - 12}$. Label horizontal and vertical asymptotes, and identify any discontinuities, x-intercepts, and the y-intercept if they exist. Describe the end behavior of the function.**



Horizontal asymptote at $y = 0$; vertical asymptote at $x = -4$; y-intercept at $\left(0, \frac{1}{4}\right)$; no x-intercepts; discontinuity at $x = 3$; as $x \to \pm\infty$, $y \to 0$.

2. **Does the graph of the function $f(x) = \frac{x^2 - 8x - 9}{x + 1}$ have a vertical asymptote or a point missing at $x = -1$? Explain your reasoning, and support your answer numerically.**

The graph of f has a point missing at $x = -1$. I knew this because the function can simplify to $f(x) = x - 9$, $x \neq -1$, which means the graph looks like the line $y = x - 9$ at every x-value except $x = -1$. This can also be seen by analyzing a table of values.

x	-1.01	-1.1	-0.99	-0.9
$f(x)$	-10.01	-10.1	-9.99	-9.9

As $x \to -1$, $f(x) \to -10$, which means the graph does not have a vertical asymptote at $x = -1$.

Problem Set Sample Solutions

1. **List all of the key features of each rational function and its graph, and then sketch the graph showing the key features.**

   a. $y = \frac{x}{x-1}$

   D: set of all real numbers except $x = 1$

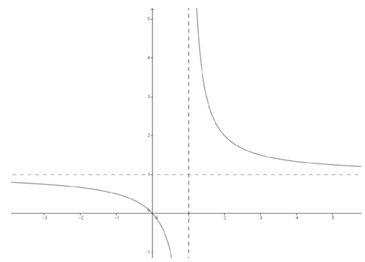
   Vertical asymptote: $x = 1$

   End behavior: As $x \to \infty$, $y \to 1$.

   Horizontal asymptote: $y = 1$

   x-intercept: $(0, 0)$

   y-intercept: $(0, 0)$



   b. $y = \frac{x^2 - 7x + 6}{x^2 - 36}$

   D: set of all real numbers except $x = 6$ *and* $x = -6$

   Equivalent function: $f(x) = \frac{x-1}{x+6}$ *with* $x \neq 6$

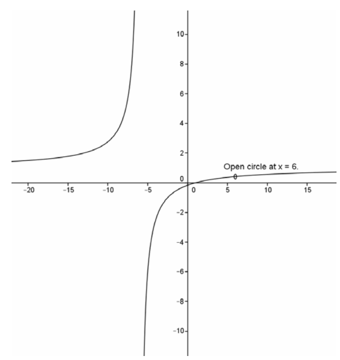
   Vertical asymptote: $x = -6$

   End behavior: As $x \to \infty$, $y \to 1$.

   Horizontal asymptote: $y = 1$

   x-intercept: $(0, 0)$

   y-intercept: $(0, 0)$



   c. $y = \frac{x^3 - 3x^2 - 10x}{x^2 + 8x - 65}$

   D: set of all real numbers except $x = -13$ *and* $x = 5$

   Equivalent function: $f(x) = \frac{x(x+2)}{x+13}$ *with* $x \neq 5$

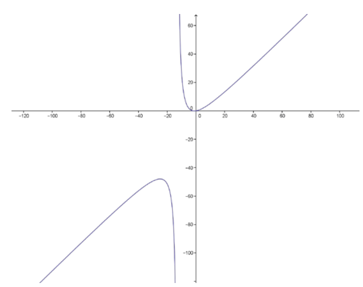
   Vertical asymptote: $x = -13$

   End behavior: As $x \to \infty$, $y \to \infty$.

   Horizontal asymptote: None—there is a slant asymptote.

   x-intercepts: $(0, 0)$, $(-2, 0)$

   y-intercept: $(0, 0)$



d. $y = \frac{3x}{x^2 - 1}$

D: set of all real numbers except $x = -1$ and $x = 1$

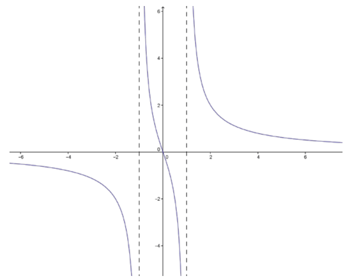
Vertical asymptotes: $x = -1$ and $x = 1$

End behavior: As $x \to \infty$, $y \to 0$.

Horizontal asymptote: $y = 0$

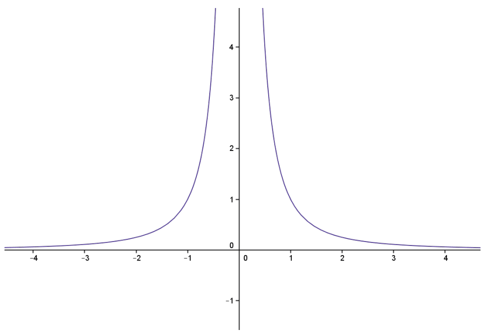
x-intercept: $(0, 0)$

y-intercept: $(0, 0)$

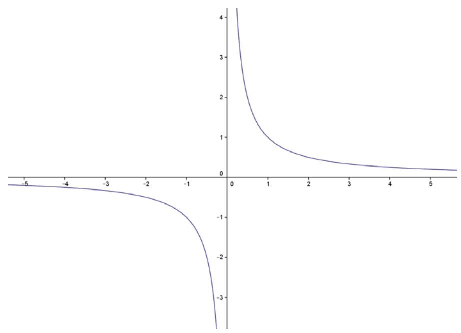


2. Graph $y = \frac{1}{x^2}$ and $y = \frac{1}{x}$. Compare and contrast the two graphs.

$y = \frac{1}{x^2}$



$y = \frac{1}{x}$



Neither graph has any intercepts. The graphs have the same vertical asymptote ($x = 0$) and horizontal asymptote ($y = 0$). The difference is that the graph of $y = \frac{1}{x^2}$ is always positive while the graph of $y = \frac{1}{x}$ is negative for x-values less than 0.

Extension:

3. Consider the function $f(x) = \frac{x^3 + 1}{x}$.

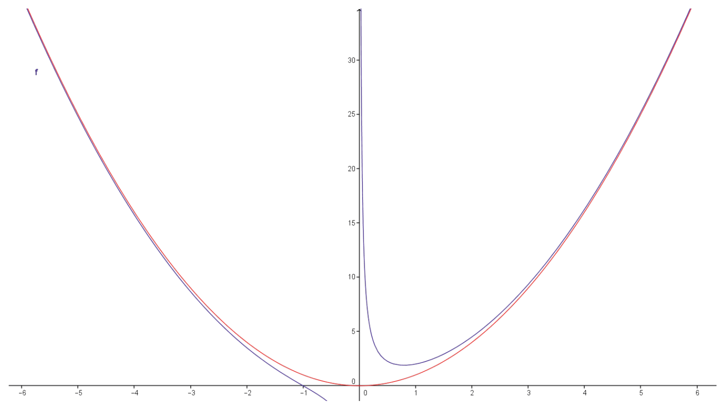
a. Use the distributive property to rewrite f as the sum of two rational functions g and h.

$f(x) = x^2 + \frac{1}{x}$; $g(x) = x^2$; $h(x) = \frac{1}{x}$

b. What is the end behavior of g? What is the end behavior of h?

As $x \to \pm\infty$, $g(x) \to \infty$. As $x \to \pm\infty$, $h(x) \to 0$.

c. Graph $y = f(x)$ and $y = x^2$ on the same set of axes. What do you notice?



d. Summarize what you have discovered in parts (b) and (c).

As x increases without bound, the graph of f approaches the graph of g, and similarly, as x decreases without bound.

4. Number theory is a branch of mathematics devoted primarily to the study of integers. Some discoveries in number theory involve numbers that are impossibly large such as Skewes' numbers and Graham's number. One Skewes' number is approximately $e^{\left(e^{\left(e^{79}\right)}\right)}$, and Graham's number is so large that to even write it requires 64 lines of writing with a new operation (one that can be thought of as the shortcut for repeated exponentiation). In fact, both of these numbers are so large that the decimal representation of the numbers would be larger than the known universe and dwarf popular large numbers such as googol and googolplex (10^{100} and $10^{(10^{100})}$, respectively). These large numbers, although nearly impossible to comprehend, are still not at the "end" of the real numbers, which have no end. Consider the function $f(x) = x^2 - 10^{100}$.

a. Consider only positive values of x; how long until $f(x) > 0$?

$$x^2 > 10^{100}$$
$$x > (10^{100})^{1/2}$$
$$x > 10^{50}$$

b. If your answer to part (a) represented seconds, how many billions of years would it take for $f(x) > 0$? (Note: One billion years is approximately 3.15×10^{16} seconds.) How close is this to the estimated geological age of the earth (4.54 billion years)?

$$10^{50} \div (3.15 \times 10^{16}) \approx 3.17 \times 10^{30}$$

It would take 3.17 thousand billion billion billion years. This is about 3.17 thousand billion billion billion years older than the earth.

c. **Number theorists frequently only concern themselves with the term of a function that has the most influence as $x \to \infty$. Let $f(x) = x^3 + 10x^2 + 100x + 1000$, and answer the following questions.**

i. **Fill out the following table:**

x	$f(x)$	x^3	$\frac{x^3}{f(x)}$	$10x^2$	$\frac{10x^2}{f(x)}$	$100x$	$\frac{100x}{f(x)}$	$\frac{1000}{f(x)}$
0	$1,000$	0	0	0	0	0	0	1.00
10	$4,000$	$1,000$	0.25	$1,000$	0.25	$1,000$	0.25	0.25
100	$1,111,000$	$1,000,000$	0.90	$100,000$	0.09	$10,000$	0.01	0.00
$1,000$	$1,010,101,000$	$1,000,000,000$	0.99	$10,000,000$	0.01	$100,000$	0.00	0.00

ii. **As $x \to \infty$, which term of f dominates the value of the function?**

The term x^3 grows to dominate the function. By $x = 1000$, x^3 comprises 99% of the function's value.

iii. **Find $g(x) = \frac{f(x)}{x}$. Which term dominates g as $x \to \infty$?**

$$g(x) = x^2 + 10x + 100 + \frac{1000}{x}$$

Again, the leading term dominates the function as $x \to \infty$.

d. **Consider the formula for a general polynomial, $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for real numbers a_i, $0 \le i \le n$. Which single term dominates the value of f as $x \to \infty$?**

The leading term dominates the function as $x \to \infty$.

Lesson 15: Transforming Rational Functions

Student Outcomes

- Students graph transformations of functions in the form $f(x) = \frac{1}{x^n}$.

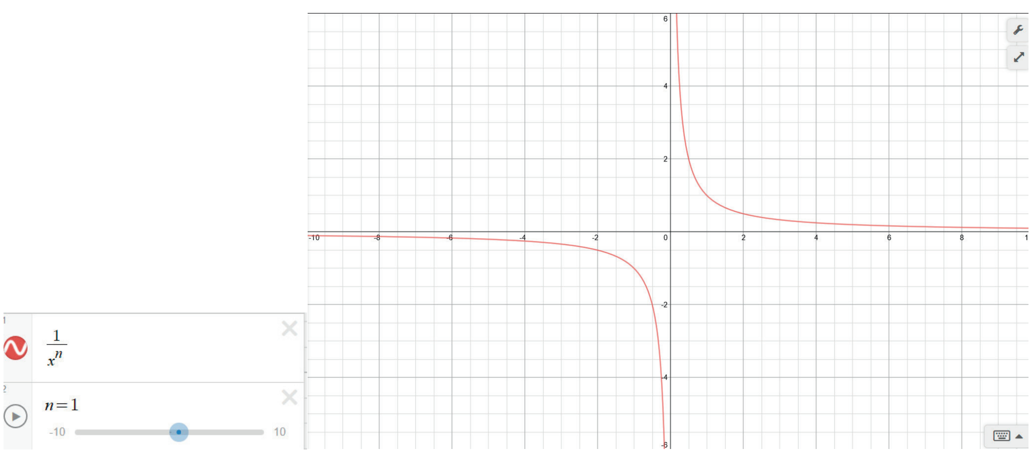
Lesson Notes

This lesson follows the pattern used in both Algebra I and Algebra II: Introduce a new type of function, and then transform it. In this way, a coherent connection is being made between transformational geometry and functions. Students have done this with a variety of functions including linear, quadratic, cubic, square root, cube root, exponential, logarithmic, and trigonometric functions. In this lesson, students explore transformations of the functions $f(x) = \frac{1}{x^n}$ with the main emphasis being on the functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. Students need to recall how to rewrite rational expressions in an equivalent form, which they learned in Algebra II (see Module 1 Lesson 22). Graphing is not as precise as it was in the previous two lessons. The focus here is on the transformations. Students should show the correct vertical and horizontal asymptotes, but it is not necessary for them to show exact points on the graph.

Classwork

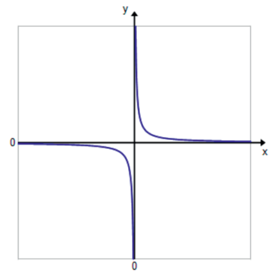
Exploratory Challenge/Exercises 1–2 (7 minutes)

Give students time to work on the Exploration in groups. Students can use graphing calculators and change the value of n in the function $f(x) = \frac{1}{x^n}$, or if computers are available, students could use a free online graphing calculator. The graph shown below can be created ahead of time and saved, or students can generate it very quickly. They can then use the slider to explore various values of n. Make sure that students have the correct graphs before they move on to the next set of exercises. If technology is not available, students can graph the functions by hand. Assign different groups different values of n in order to produce a variety of functions.

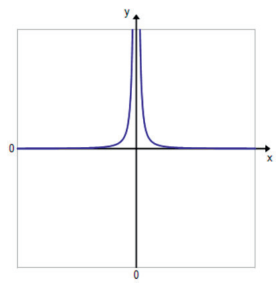


Exploratory Challenge/Exercises 1–2

1. **Sketch the general shape of the graph of the function $f(x) = \frac{1}{x^n}$ for $n > 0$ when n is an odd number.**



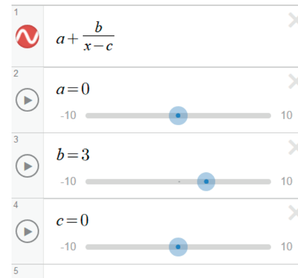
2. **Sketch the general shape of the graph of the function $f(x) = \frac{1}{x^n}$ for $n > 0$ when n is an even number.**



MP.7 & MP.8

Scaffolding:

If students are struggling, use dynamic graphing software to explore the effects of the various parameters on the graph of the function.



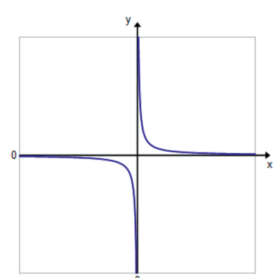
As a challenge opportunity, ask students to explain why they believe these patterns hold.

Exercises 3–5 (10 minutes)

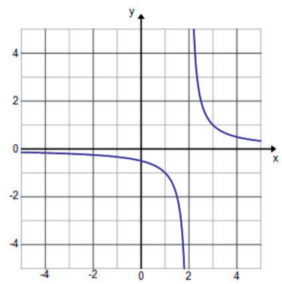
Allow students time to work in groups on the next set of exercises. Consider letting students use a graphing calculator or software to check their graphs, but make sure they are sketching the graphs first without using technology.

Exercises 3–5

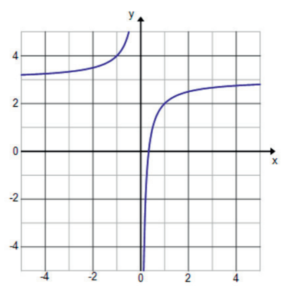
3. **Sketch the graph of the function $f(x) = \frac{1}{x}$. Then, use the graph of f to sketch each transformation of f showing the vertical and horizontal asymptotes.**



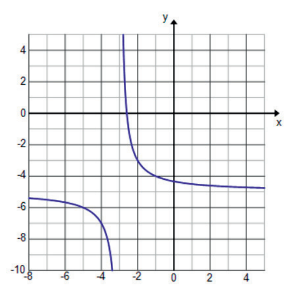
a. $g(x) = \frac{1}{x-2}$



b. $h(x) = -\frac{1}{x} + 3$



c. $k(x) = \frac{2}{x+3} - 5$

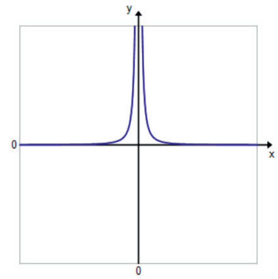


MP.7 & MP.8

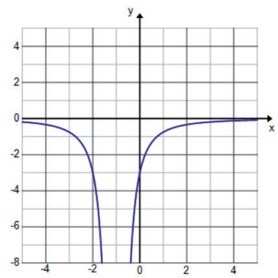
4. **Use your results from Exercise 3 to make some general statements about graphs of functions in the form $f(x) = a + \frac{b}{x-c}$. Describe the effect that changing each parameter $a, b,$ and c has on the graph of f.**

a represents the vertical shift. A positive value of a shifts the graph up a units, and a negative value shifts the graph down a units. The horizontal asymptote is $y = a$. c represents the horizontal shift. A positive value of c shifts the graph right c units, and a negative value shifts the graph left c units. The vertical asymptote is $x = c$. b represents the vertical scaling. For values of b such that $|b| > 1$, the graph stretches vertically. For values of b such that $0 < |b| < 1$, the graph compresses vertically. When b is negative, the graph reflects across the x-axis.

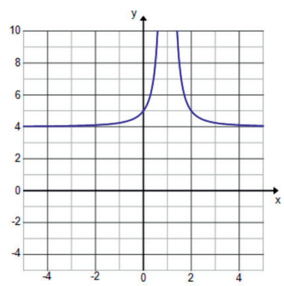
5. **Sketch the graph of the function $f(x) = \frac{1}{x^2}$. Then, use the graph of f to sketch each transformation of f showing the vertical and horizontal asymptotes.**



a. $g(x) = -\frac{3}{(x+1)^2}$



b. $h(x) = \frac{1}{(x-1)^2} + 4$



- Since we were not plotting any precise points, what was the easiest way to track the transformations?
  - *By shifting the vertical and horizontal asymptotes. The horizontal shift tells us the new vertical asymptote, and the vertical shift tells us the new horizontal asymptote.*

Example 1 (5 minutes)

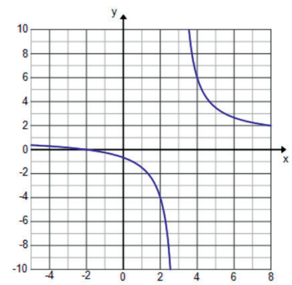
Lead students through a review of rewriting a rational expression into an equivalent form. Then, have students sketch the graph using transformations of functions.

- Can we use transformations to graph f in the form given?
  - *No. It is not in a form where we can use the graph of $y = \frac{1}{x}$ to graph f.*
- Could we rewrite f into an equivalent form? Lead students to the idea that we want the function to be in the form $f(x) = a + \frac{b}{x-c}$.

Example 1

Graph the function $f(x) = \frac{x+2}{x-3}$ using transformations of the graph of $y = \frac{1}{x}$.

$f(x) = \frac{x-3+5}{x-3} = 1 + \frac{5}{x-3}$



Exercises 6–13 (15 minutes)

Allow students time to work in groups on the next set of exercises. Students may need to be reminded about completing the square on Exercise 9.

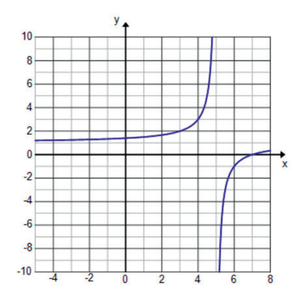
Exercises 6–13

Sketch each function by using transformations of the graph of $y = \frac{1}{x}$ or the graph of $y = \frac{1}{x^2}$. Explain the transformations that are evident in each example.

6. $f(x) = \frac{x-7}{x-5}$

   $f(x) = 1 - \frac{2}{x-5}$

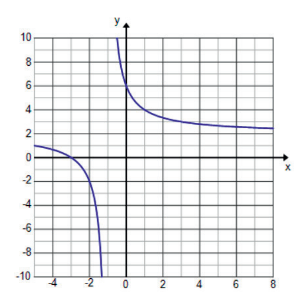
   The graph is shifted right 5 units.



7. $f(x) = \frac{2x+6}{x+1}$

   $f(x) = \frac{x+1+x+1+4}{x+1} = 1 + 1 + \frac{4}{x+1} = 2 + \frac{4}{x+1}$

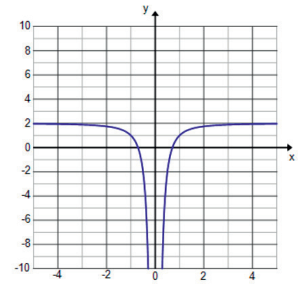
   The graph is shifted left 1 unit and up 2 units.



8. $f(x) = \dfrac{2x^2 - 1}{x^2}$

$f(x) = 2 - \dfrac{1}{x^2}$

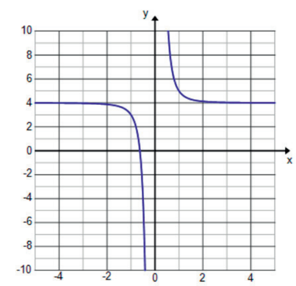
The graph is shifted up 2 units, and both branches are going down.



9. $f(x) = \dfrac{1 + 4x^3}{x^3}$

$f(x) = \dfrac{1}{x^3} + 4$

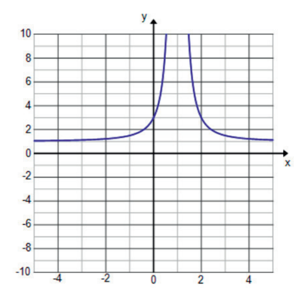
The graph is shifted up 4 units.



10. $f(x) = \dfrac{x^2 - 2x + 3}{(x-1)^2}$

$f(x) = \dfrac{(x^2 - 2x + 1) + 3 - 1}{(x - 1)^2} = \dfrac{(x - 1)^2 + 2}{(x - 1)^2} = 1 + \dfrac{2}{(x - 1)^2}$

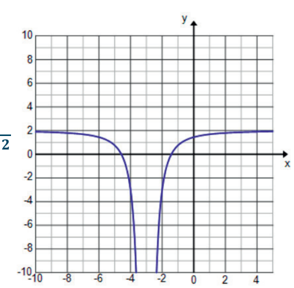
The graph is shifted right 1 unit and up 1 unit. Both branches go up.



11. $f(x) = \dfrac{2x^2 + 12x + 13}{(x+3)^2}$

$f(x) = \dfrac{2(x^2 + 6x + 9) + 13 - 18}{(x + 3)^2} = \dfrac{2(x + 3)^2 - 5}{(x + 3)^2} = 2 - \dfrac{5}{(x + 3)^2}$

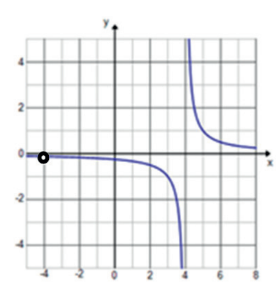
The graph is shifted left 3 units and up 2 units. Both branches go down.



12. $f(x) = \dfrac{x + 4}{x^2 - 16}$

$f(x) = \dfrac{1}{x - 4}, x \neq -4$

The graph is shifted right 4 units, and there is a value missing at $x = -4$.

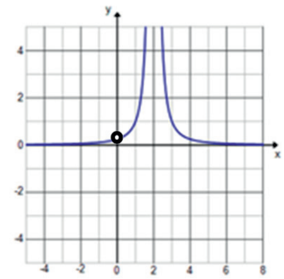


13. $f(x) = \dfrac{x}{x^3 - 4x^2 + 4x}$

$f(x) = \dfrac{1}{(x-2)^2}, x \neq 0$

The graph is shifted right 2 units, and there is a value missing at $x = 0$.

Both branches go up.



MP.7

- What technique did you use to rewrite the function in Exercise 6 so that transformations could be used to sketch the graph? (Note: Show different approaches.)
  - *I used long division. I expanded $2x + 6$ into $x + 1 + x + 1 + 4$.*
- What technique did you use to rewrite the function in Exercise 9?
  - *I completed the square in the numerator so that it was in the form $(x-1)^2 + 2$ and then separated the expression into two terms.*
- How did Exercises 11 and 12 differ from the other exercises?
  - *They could both be rewritten as a simpler function by reducing the common factors in the numerator and denominator.*

Closing (3 minutes)

- Describe the effect changing each parameter $a, b, and\ c$ has on the graph of $f(x) = a + \dfrac{b}{x-c}$.
  - *a represents the vertical shift. A positive value of a shifts the graph up a units, and a negative value shifts the graph down a units. The horizontal asymptote is $y = a$. c represents the horizontal shift. A positive value of c shifts the graph right c units, and a negative value shifts the graph left c units. The vertical asymptote is $x = c$. b represents the vertical scaling. For values of b such that $|b| > 1$, the graph stretches vertically. For values of b such that $0 < |b| < 1$, the graph compresses vertically. When b is negative, the graph reflects across the x-axis.*
- Describe the effect of the parameter n on the graph of $f(x) = \dfrac{1}{x^n}$.
  - *When n is odd, both branches go in opposite directions toward the vertical asymptote. When n is even, both branches go in the same direction toward the vertical asymptote.*
- How did the approach we took to graphing rational functions in this lesson differ from the previous two lessons?
  - *We used transformations rather than an analysis of the key features of the graph. We graphed less precisely, using transformations rather than exact points on the graph.*

Exit Ticket (5 minutes)

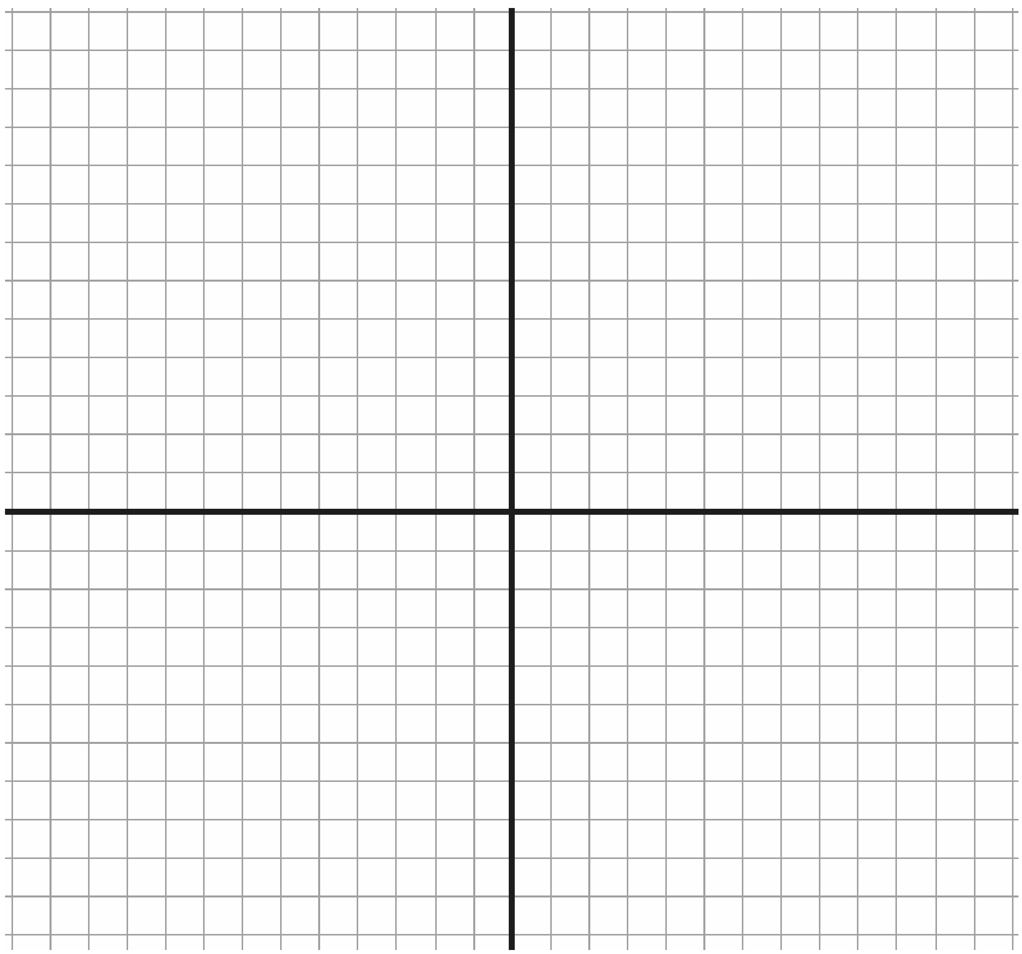
Name ____ Date ____________________

Lesson 15: Transforming Rational Functions

Exit Ticket

Sketch the graph of the function given below by using transformations of $y = \frac{1}{x^n}$. Explain which transformations you used and how you identified them.

$$y = \frac{3x - 7}{x - 3}$$



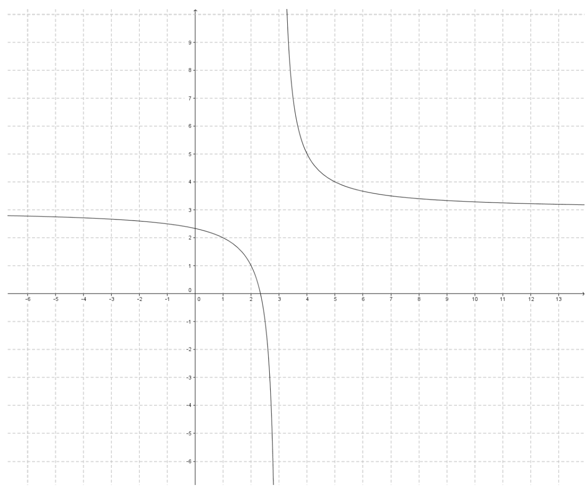
Exit Ticket Sample Solutions

Sketch the graph of the function given below by using transformations of $y = \frac{1}{x^n}$. Explain which transformations you used and how you identified them.

$$y = \frac{3x-7}{x-3}$$

$$y = \frac{3x-9+2}{x-3} = 3 + \frac{2}{x-3}$$

By rewriting the function into the form $a + \frac{b}{x-c}$, I saw that the graph was shifted right 3 units, stretched vertically by a factor of 2, and shifted up 3 units.



Problem Set Sample Solutions

1. Write each function so that it appears to be a transformation of $y = \frac{1}{x^n}$. Then, explain how the graph of each function relates to the graph of $y = \frac{1}{x^n}$.

   a. $y = \frac{5x-8}{x+2}$

   $$y = \frac{5x+10-18}{x+2} = 5 - \frac{18}{x+2}$$

   The graph would be the graph of $y = \frac{1}{x}$ shifted left 2 units, stretched vertically by a scale factor of 18, reflected across the x-axis, and shifted up 5 units.

   b. $y = \frac{2x^3-4}{x^3}$

   $$y = 2 - \frac{4}{x^3}$$

   The graph would be the graph of $y = \frac{1}{x^3}$ stretched vertically by a scale factor of 4, reflected across the x-axis, and shifted up 2 units.

c. $y = \dfrac{x^2 - 4x + 8}{(x-2)^2}$

$$\begin{aligned} y &= \frac{x^2 - 4x + 4 + 4}{(x-2)^2} \\ &= \frac{(x-2)^2 + 4}{(x-2)^2} \\ &= 1 - \frac{4}{(x-2)^2} \end{aligned}$$

The graph would be the graph of $y = \frac{1}{x^2}$ shifted right 2 units, stretched vertically by a scale factor of 4, reflected across the x-axis, and shifted up 1 unit.

d. $y = \dfrac{3x - 12}{x^2 - 16}$

$$\begin{aligned} y &= \frac{3(x-4)}{(x-4)(x+4)} \\ &= \frac{3}{x+4};\ x \neq 4 \end{aligned}$$

The graph would be the graph of $y = \frac{1}{x}$ shifted left 4 units and stretched vertically by a scale factor of 3. The point at $x = 4$ would be missing from the graph.

e. $y = \dfrac{2x^2 + 16x + 25}{x^2 + 8x + 16}$

$$\begin{aligned} y &= \frac{2x^2 + 16x + 25}{x^2 + 8x + 16} \\ &= \frac{2x^2 + 16x + 25}{(x+4)^2} \\ &= \frac{2(x^2 + 8x + 16) + 25 - 32}{(x+4)^2} \\ &= 2 - \frac{7}{(x+4)^2} \end{aligned}$$

The graph would be the graph of $y = \frac{1}{x^2}$ shifted left 4 units, stretched vertically by a scale factor of 7, reflected across the x-axis, and shifted up 2 units.

2. **For each function in Problem 1, state how the horizontal and vertical asymptotes are affected from the original graph of $y = \frac{1}{x^n}$.**

   a. The horizontal asymptote is moved up 5; the vertical asymptote is moved 2 to the left.

   b. The horizontal asymptote is moved up 2; the vertical asymptote is unchanged.

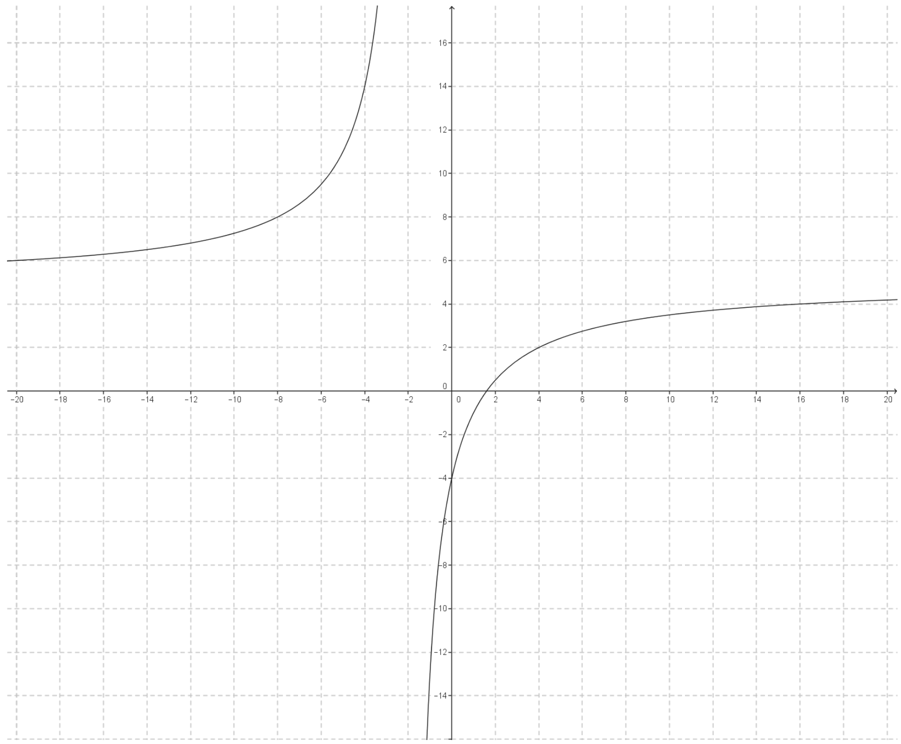
   c. The horizontal asymptote is moved up 1; the vertical asymptote is moved to the right 2.

   d. The horizontal asymptote is unchanged; the vertical asymptote is moved to the left 4.

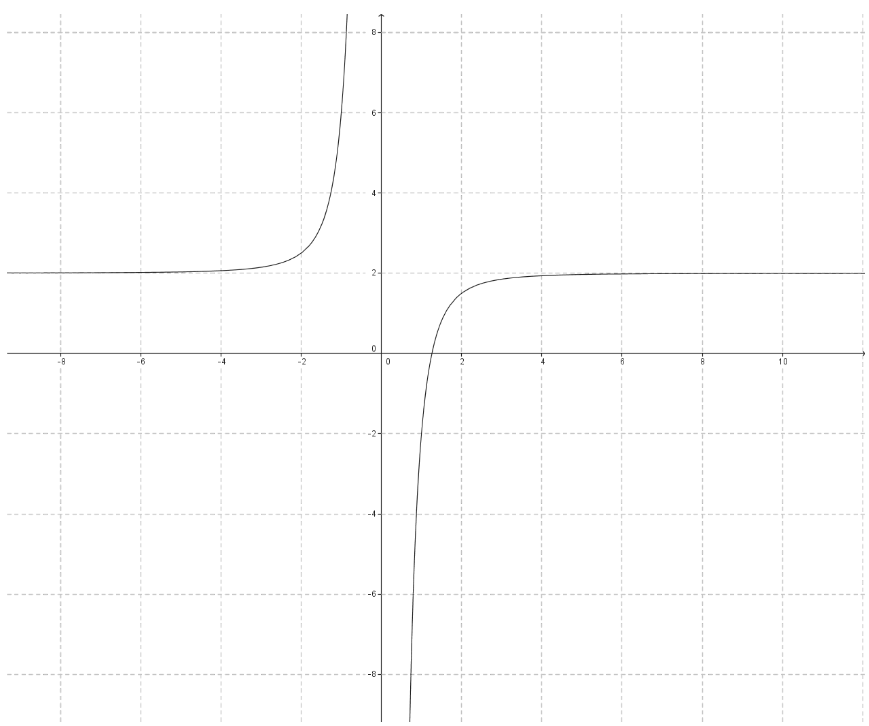
   e. The horizontal asymptote is moved up 2; the vertical asymptote is moved to the left.

3. **Sketch a picture of the graph of each function in Problem 1.**

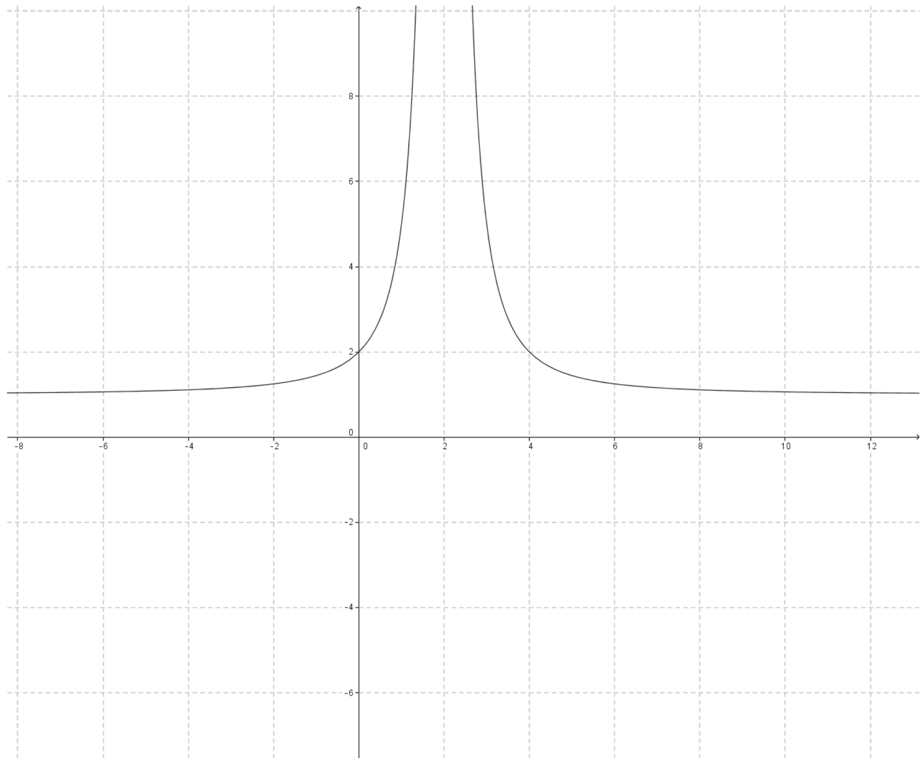
a.



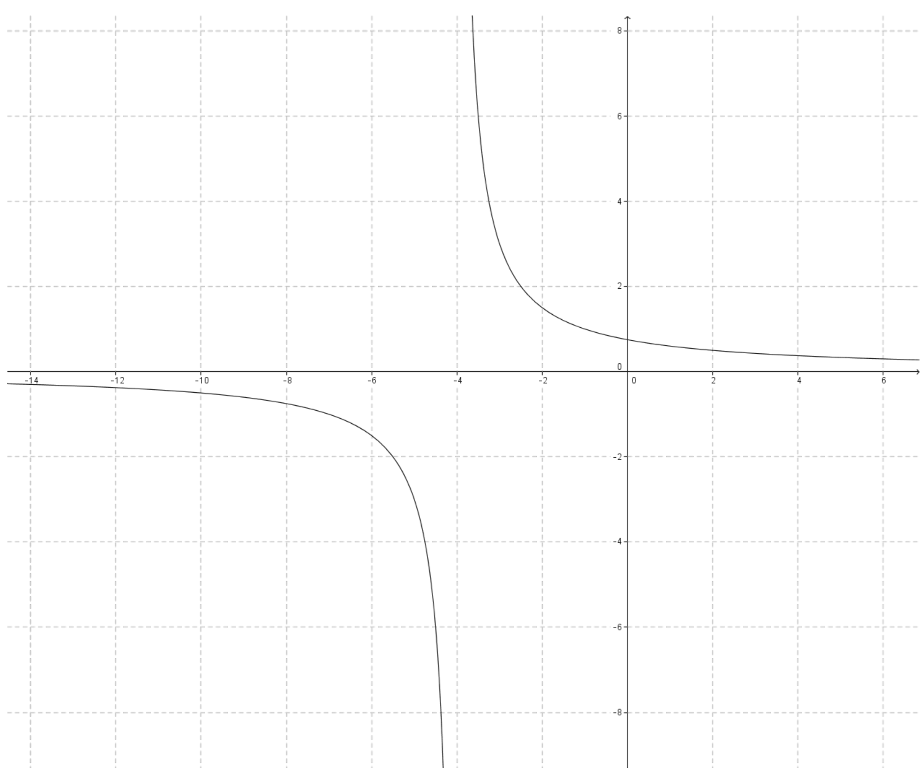
b.



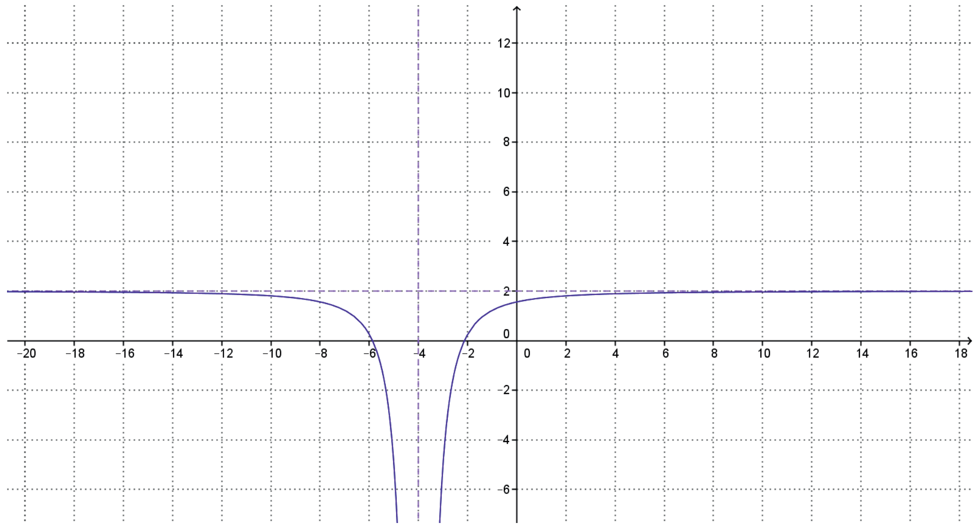
c.



d.



e.



4. **What are some indicators whether or not a rational function can be expressed as a transformation of $y = \frac{1}{x^n}$?**

   If the variables in the numerator can be expressed as a part of a multiple of the denominator, then the function is able to be expressed as a transformation. If it cannot, then there is a variable term in the numerator that cannot be removed that is the result of a composition or product of functions and not from a transformation.

5. **Write an equation for a function whose graph is a transformation of the graph $y = \frac{1}{x}$. The graph has been shifted right 2 units, stretched vertically by a factor of 2, and shifted down 3 units.**

   $y = \frac{2}{x - 2} - 3$

Lesson 16: Function Composition

Student Outcome

- Students compose functions and describe the domain and range of the compositions.

Lesson Notes

In previous courses, students have been introduced to functions as relationships between sets of numbers where each input in the domain is assigned a unique output in the range. In this lesson, students explore functions and their compositions, including situations where the sets representing the inputs and outputs may not be numerical. They determine whether or not compositions are defined, and find the composition of functions in real-world contexts.

Classwork

Opening (7 minutes)

Students should be familiar with the term *function* and its characteristics in a mathematical context. This lesson broadens students' understanding of functions to include functions on nonnumerical sets, and it expands on students' understanding of performing operations on functions to include composing functions. An example addressing competitive free diving exposes students to a real-world situation where composing functions could be used to solve problems. Explain to students that free diving is a process by which divers descend a given distance in the water and then swim to the surface, all without any breathing aid, which can be dangerous to competitors because the rapid change in pressure the divers experience as they descend can cause nitrogen bubbles to form in their capillaries, inhibiting blood flow. Pressure is measured in units of atmospheres (atm); one atmosphere is the mean atmospheric pressure at sea level, which is $101{,}325$ newtons per square meter. Students should analyze the tables provided and respond to the prompt. Several students can share their reflections briefly in a whole-class setting:

	Depth of Free Diver During Descent									
s time of descent, in seconds	0	20	40	60	80	100	120	140	160	180
d depth of diver, in meters	0	15	32	44	65	79	90	106	120	133

	Atmospheric Pressure and Ocean Depth									
d depth of diver, in meters	0	10	20	30	40	50	60	70	80	90
p pressure on diver, in atmospheres	1	2	3	4	5	6	7	8	9	10

Scaffolding:

- Call out times of descent from the first table, and have students call out the corresponding depths.
- Call out depths from the second table, and have students call out the corresponding pressures.

- Based on the information you have been presented about free diving and the data in the tables, how might medical researchers use the concept of functions to predict the atmospheric pressure on a free diver during his descent?

Example 1 (10 minutes)

This example provides students with the opportunity to determine whether the tables from the opening scenario have the characteristics of functions and to interpret the domain and range in context. Students also explore the composition of two functions, which prepares them to compose functions and evaluate compositions of functions later in the lesson. The example should be completed as part of a teacher-led discussion.

- What do you recall about functions between numerical sets?
  - *Answers should address that for each input in the domain, there is exactly one output in the range; there may be restrictions on the domain and/or range.*
- In what contexts have you analyzed sets of numbers to determine whether the relationship between them represents a function?
  - *Answers may vary but might include the following: mapping diagrams to determine if each input in the domain is assigned to only one output in the range; writing all the inputs and corresponding outputs as ordered pairs and determining if all the x-values are unique; plotting these ordered pairs on a coordinate plane; determining whether the graph passes the vertical line test.*
- Do the conditions for functions appear to hold for the table relating a free diver's descent time and depth? How about for the table relating depth and pressure? Explain.
  - *Yes for both tables. In the first table, each time entered corresponds to exactly one depth. For the second table, each depth corresponds to exactly one pressure.*
- The first table represents depth of the diver, in meters, as a function of the time of descent, in seconds. The independent variable (the input) is time, and the dependent variable (the output) is depth.
- The second table represents the pressure on the diver, in atmospheres, as a function of the depth of the diver, in meters. The independent variable (the input) is depth, and the dependent variable (the output) is pressure.
- In the first table, what do the domain and range represent?
  - *Domain is time spent descending, in seconds, and range is depth of descent, in meters.*
- Why can't these values be negative?
  - *It would not make sense for a diver to descend for a negative number of seconds or to a depth that is above the ocean's surface.*
- What other restrictions could we place on the domain and range for this function?
  - *There is some reasonable upper limit that represents the amount of time a diver can spend descending and a maximum depth the diver can reach.*
- Define and discuss the domain and range of the function represented by the second table.
  - *The domain is the depth of the descending diver in meters, which cannot be negative and has some real-number maximum value that corresponds with the greatest depth a diver can reach. The range is the pressure applied to the diver. This cannot be less than 1 atmosphere (atm), the pressure at sea level, and has a maximum value that corresponds to the maximum depth of the descending diver.*

- How can we use the tables to determine the depth of a diver who has descended for 80 seconds? What is that depth?
  - *Find the column in the first table where $s = 80$, and find the corresponding value of d, which is 65.*
- How can we represent this relationship using function notation?
  - $f(80) = 65$

MP.2

- So, what would $f(20) = 15$ represent in context?
  - *The depth of a diver who has descended for 20 seconds is 15 meters.*
- How can we use the tables to determine the pressure applied to a diver who has descended 40 meters, and what is it?
  - *Find the column in the second table where $d = 40$, and find the corresponding value of p, which is 5.*
- How can we represent this relationship using function notation, and what does it represent in context?
  - $p(40) = 5$*; a diver has a pressure of 5 atmospheres applied to her at a depth of 40 meters.*
- Now how could we use the tables to find the pressure applied to a diver 120 seconds into the descent?
  - *Use the first table to find the depth of the diver at 120 seconds, which is 90 meters, and then use the second table to find the pressure at 90 meters, which is 10 atmospheres.*
- How can we explain this process in terms of functions?
  - *Evaluate $f(120)$ and use this as the input for the function $p = g(d)$.*
- And how can the overall process be represented using function notation?
  - $p = g\big(f(120)\big)$
- Students may struggle with this new concept. Practice with many different examples.
- Pause to explain that students have composed the functions in the tables to create a new function of pressure on the diver as a function of time of descent.

Example 1

Consider the tables from the opening scenario.

Depth of Free Diver During Descent									
s time of descent, in seconds	20	40	60	80	100	120	140	160	180
d depth of diver, in meters	15	32	44	65	79	90	106	120	133

Atmospheric Pressure and Ocean Depth									
d depth of diver, in meters	10	20	30	40	50	60	70	80	90
p pressure on diver, in atm	2	3	4	5	6	7	8	9	10

> *Scaffolding:*
> Have advanced students answer parts (d)–(f) only.

a. Do the tables appear to represent functions? If so, define the function represented in each table using a verbal description.

Both tables appear to represent functions because for each input in the domain, there is exactly one output.

In the first table, the depth of the diver is a function of the time spent descending.

In the second table, the pressure on the diver is a function of the diver's depth.

b. What are the domain and range of the functions?

For the first table, the domain and range are nonnegative real numbers.

For the second table, the domain is nonnegative real numbers, and the range is real numbers greater than or equal to 1.

c. Let's define the function in the first table as $d = f(s)$ and the function in the second table as $p = g(d)$. Use function notation to represent each output, and use the appropriate table to find its value.

i. Depth of the diver at 80 seconds

$d = f(80) = 65$. ***After 80 seconds, the diver has descended 65 meters.***

ii. Pressure on the diver at a depth of 60 meters

$p = g(60) = 7$. ***At a depth of 60 meters, there are 7 atmospheres of pressure on the diver.***

d. Explain how we could determine the pressure applied to a diver after 120 seconds of descent.

We could use the first table to determine the depth that corresponds to a descent time of 120 seconds and then use the second table to find the pressure that corresponds to this depth.

e. Use function notation to represent part (d), and use the tables to evaluate the function.

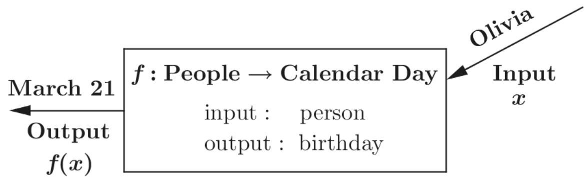
$g\big(f(120)\big) = g(90) = 10$

f. *Describe the output from* part *(e) in context.*

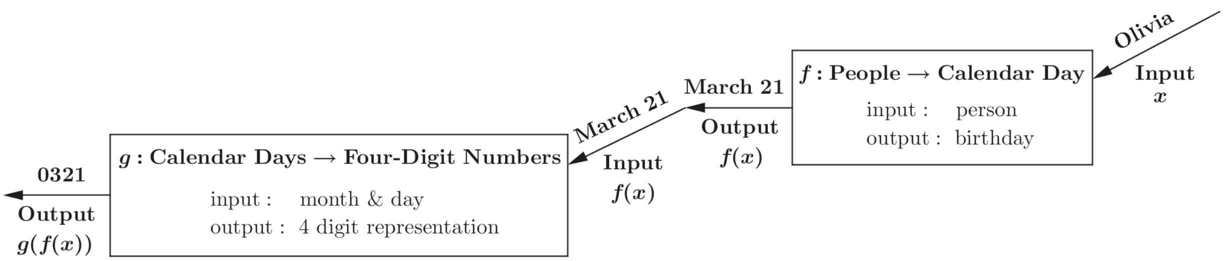
The pressure applied to a diver 120 seconds into a descent is 10 atmospheres.

Discussion (7 minutes): Composing Functions

- So, now we have reviewed the necessary conditions for a relationship between sets to be considered a function and examined a composition of two functions. Let's explore a composition in more depth. Suppose that we have two functions f and g. Function f has domain of all people, range days of the year, and assigns each person to the month and year of her birthday. For example, $f(\text{Abraham Lincoln}) = \text{Feb. } 12$. Function g has domain of days of the year and range of four-digit numbers, including those that begin with 0. Function g assigns each day of the year to the four-digit representation MMDD, representing the two-digit month (01 for January, 12 for December) and two-digit day. For example, $g(\text{Feb. } 12) = 0212$. If we imagine that function f represents a box with an "in" chute and an "out" chute, inputs from the domain would enter through the "in" chute, and the corresponding outputs from the range would exit from the "out" chute.



- Describe what is occurring in the diagram.
  - *The function rule, f is being applied to the input, Olivia, to produce the output, March 21.*
- And how do we generally notate inputs and outputs for function rule f?
  - *The input is x, and the output is $f(x)$.*
- Now if we want to compose the two functions, we can imagine placing another box g where the outputs of f are fed into the "in" chute of g, and the function rule for g is applied, producing an output $g\big(f(x)\big)$:



- Explain the process illustrated by the diagram.
  - *Answers might vary but should address that "Olivia" was the input into function f, which produced output "March 21"; "March 21", in turn, became the input into function g, which produced the output* "0321."
- So, for our function composition, what is the relationship between the inputs and outputs for f and g?
  - *The output of f must be a valid input of g.*
- And what is the overall effect of the process on the initial input x?
  - *The function rule f is applied to x, and then g is applied to $f(x)$.*
- Let's formalize our description of a function composition:

> If f and g are two functions so that the range of f lies within the domain of g, that is, $f\colon X \to Y$ and $g\colon Y \to Z$ for sets X, Y, and Z, then the *composite function* of f and g, denoted by $g \circ f$, is the combined function f followed by g.

- What other notation can we use to represent the composition $(g \circ f)(x)$?
  - $g\big(f(x)\big)$
- How would we interpret $(g \circ f)(x)$ in our people and birthdays example?
  - *The value of $(g \circ f)(x)$ is the four-digit number that corresponds to the birthday of person x.*

- And in what order are the functions applied?
  - *From right to left; first, the person is assigned to his birthday and then, the birthday is assigned to the four-digit number.*

Example 2 (5 minutes)

Example 2 below provides students with the opportunity to examine composition of two functions. Students compose functions and interpret them in context, including determining whether compositions are defined.

- How do we perform the composition $f \circ g$? Interpret the process in the context of the example.
  - *First, we apply the function rule g to an element of the domain, and then, we apply f to the result. For this situation, that means we assign a person to her favorite animal, and then we assign that animal to its number of legs.*
- So, what process would be applied to compute $g \circ f$?
  - *We would apply the function rules from right to left. This means that we would assign an animal to its number of legs, but then the resulting number is not a valid input into the function g. The range of f is not contained within the domain of g.*
- What conclusions can we draw about performing compositions on functions?
  - *The function rules should be applied from right to left (inside to outside), with the output of each function rule representing the input of the next function applied; not all compositions make sense in context.*
- So, if $f(x) = 2x$ and $g(x) = x + 4$, how would you find $f\big(g(3)\big)$? $(g \circ f)(x)$?
  - *To find $f\big(g(3)\big)$, evaluate $g(3)$, and apply f to the result, so $f\big(g(3)\big) = f(3 + 4) = f(7) = 2(7) = 14$; to find $(g \circ f)(x)$, evaluate $f(x)$, and apply g to the result, so $(g \circ f)(x) = g(2x) = 2x + 4$.*

Example 2

Consider these functions:

f: **Animals → Counting numbers**

Assign to each animal the number of legs it has.

g: **People → Animals**

Assign to each person his favorite animal.

MP.2

Determine which composite functions are defined. If defined, describe the action of each composite function.

a. $f \circ g$

Assign a person to her favorite animal, then assign the animal to its number of legs: The composite function is defined, and assign each person to the number of legs of her favorite animal

b. $f \circ f$

This composition is not defined. Function f assigns a number to an animal, but it cannot accept the number that it outputs as an input. The range of f is not contained within the domain of f.

MP.2

c. $g \circ f$

This composition is not defined. Function f assigns each animal to a number. Function g accepts only people as inputs, so it cannot accept the number output by function f. The range of f is not contained within the domain of g.

d. $f \circ g \circ g$

The composition $g \circ g$ is not defined. Function g assigns each person to an animal, but it cannot accept the animal that it outputs as an input. The range of g is not contained within the domain of g.

Exercises (8 minutes)

These exercises should be completed individually. After a few minutes, students could verify their responses with a partner. Then, selected students could share their responses with the class. Be sure that students understand the implications of Exercises 1(a) and 1(b) and that function composition is not a commutative operation.

Scaffolding:

Work through an example using different colors for the expressions representing f and g to help students visualize how the output of the first applied function represents the input for the second function.

Exercises

1. Let $f(x) = x^2$ and $g(x) = x + 5$. Write an expression that represents each composition:

a. $g(f(4))$

$g(f(4)) = g(4^2) = g(16) = (16 + 5) = 21$

b. $f(g(4))$

$f(g(4)) = f(4+5) = f(9) = 9^2 = 81$

c. $(f \circ g)(x)$

$(f \circ g)(x) = f(x+5) = (x+5)^2 = x^2 + 10x + 25$

d. $(f \circ g)(\sqrt{x+5})$

$(f \circ g)(\sqrt{x+5}) = f(\sqrt{x+5} + 5) = (\sqrt{x+5} + 5)^2 = x + 10\sqrt{x+5} + 30$

2. Suppose a sports medicine specialist is investigating the atmospheric pressure placed on competitive free divers during their descent. The following table shows the depth, d, in meters of a free diver s seconds into his descent. The depth of the diver is a function of the number of seconds the free diver has descended, $d = f(s)$.

s time of descent, s	10	35	55	70	95	115	138	160	175
d depth, m	8.1	28	45	55	76.0	91.5	110	130	145

The pressure, in atmospheres, felt on a free diver, d, is a function of his depth, $p = g(d)$.

d depth, m	25	35	55	75	95	115	135	155	175
p pressure, atm	2.4	3.5	5.5	7.6	9.6	11.5	13.7	15.5	17.6

a. **How can the researcher use function composition to examine the relationship between the time a diver spends descending and the pressure he experiences? Use function notation to explain your response.**

The function $g(f(s))$ represents the pressure experienced by a diver who has been descending for s seconds. The function f assigns a depth in meters to each time s, and the function g assigns a pressure to each depth. Then $g \circ f$ assigns a pressure to each time.

b. **Explain the meaning of $g(f(0))$ in context.**

The value of $g(f(0))$ is the pressure on a free diver at his depth 0 seconds into the descent.

MP.2

c. **Use the charts to approximate these values, if possible. Explain your answers in context.**

i. $g(f(70))$

5.5 atmospheres; this is the pressure on the free diver at his depth 70 seconds into his dive ($d = 55$ meters).

ii. $g(f(160))$

Approximately 13 atmospheres; this is the pressure on the free diver at his depth 160 seconds into his dive ($d = 130$ meters).

Closing (3 minutes)

Have students paraphrase what they understand about composing functions using bulleted statements. If time permits, have students share their statements, and encourage them to add to their own set of statements based on what is shared.

- To perform the composition $g \circ f$, apply the function rule f to an element of the domain of f, and apply g to the result.
- The composition of functions is not a commutative operation. In other words, $g \circ f \neq f \circ g$, except in rare cases when functions are inverses.
- Not all function compositions are defined.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 16: Function Composition

Exit Ticket

1. Let $f(x) = x^2$ and $g(x) = 2x + 3$. Write an expression that represents each composition:
   a. $(g \circ f)(x)$

   b. $f(f(-2))$

   c. $(f \circ g)\left(\frac{1}{x}\right)$

2. A consumer advocacy company conducted a study to research the pricing of fruits and vegetables. They collected data on the size and price of produce items, including navel oranges. They found that, for a given chain of stores, the price of oranges was a function of the weight of the oranges, $p = f(w)$.

w weight in pounds	0.2	0.25	0.3	0.4	0.5	0.6	0.7
p price in dollars	0.26	0.32	0.39	0.52	0.65	0.78	0.91

The company also determined that the weight of the oranges measured was a function of the radius of the oranges, $w = g(r)$.

r radius in inches	1.5	1.65	1.7	1.9	2	2.1
w weight in pounds	0.38	0.42	0.43	0.48	0.5	0.53

a. How can the researcher use function composition to examine the relationship between the radius of an orange and its price? Use function notation to explain your response.

b. Use the table to evaluate $f(g(2))$, and interpret this value in context.

Exit Ticket Sample Solutions

1. Let $f(x) = x^2$ and $g(x) = 2x + 3$. Write an expression that represents each composition:

   a. $(g \circ f)(x)$

   $(g \circ f)(x) = g(x^2) = 2(x^2) + 3 = 2x^2 + 3$

   b. $f(f(-2))$

   $f(f(-2)) = f(4) = 4^2 = 16$

   c. $(f \circ g)\left(\frac{1}{x}\right)$

   $(f \circ g)\left(\frac{1}{x}\right) = f\left(2\left(\frac{1}{x}\right) + 3\right) = \left(\frac{2}{x} + 3\right)^2 = \frac{4}{x^2} + \frac{12}{x} + 9$

2. A consumer advocacy company conducted a study to research the pricing of fruits and vegetables. They collected data on the size and price of produce items, including navel oranges. They found that, for a given chain of stores, the price of oranges was a function of the weight of the oranges, $p = f(w)$.

w weight in pounds	0.2	0.25	0.3	0.4	0.5	0.6	0.7
p price in dollars	0.26	0.32	0.39	0.52	0.65	0.78	0.91

The company also determined that the weight of the oranges measured was a function of the radius of the oranges, $w = g(r)$.

r radius in inches	1.5	1.65	1.7	1.9	2	2.1
w weight in pounds	0.38	0.42	0.43	0.48	0.5	0.53

   b. How can the researcher use function composition to examine the relationship between the radius of an orange and its price? Use function notation to explain your response.

   The function $f(g(r))$ represents the price of oranges as a function of the radius of the oranges.

   c. Use the table to evaluate $f(g(2))$, and interpret this value in context.

   $f(g(2)) = f(0.5) = 0.65$

   The price of oranges with a radius of 2 inches is $\$0.65$.

Problem Set Sample Solutions

1. **Determine whether each rule described represents a function. If the rule represents a function, write the rule using function notation, and describe the domain and range.**

   a. **Assign to each person her age in years.**

   Yes. f: *People* $\rightarrow$ *Numbers*

   Domain: set of all living people. Range: $\{0, 1, 2, 3, \ldots, 130\}$

   b. **Assign to each person his height in centimeters.**

   Yes. f: *People* $\rightarrow$ *Numbers*

   Domain: set of all living people. Range: $\{50, 51, \ldots 280\}$

   c. **Assign to each piece of merchandise in a store a bar code.**

   Yes. f: *Products* $\rightarrow$ *Bar codes*

   Domain: each piece of merchandise in the store. Range: {*unique bar codes*}

   d. **Assign each deli customer a numbered ticket.**

   Yes. f: *People* $\rightarrow$ *Numbered tickets*

   Domain: set of people that are waiting in the deli. Range: {*Numbered tickets*}

   e. **Assign a woman to her child.**

   No. There are many women who have more than one child and many who have no children.

   f. **Assign to each number its first digit.**

   Yes. f: *Counting numbers* $\rightarrow$ *Counting numbers*

   Domain: set of all counting numbers. Range: $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   g. **Assign each person to the city where he was born.**

   Yes. f: *People* $\rightarrow$ *Cities*

   Domain: set of all counting numbers. Range: {*Cities*} .

2. **Let L: Animal → Counting numbers**

Assign each animal to its number of legs.

F: People → Animals

Assign each person to his favorite animal.

N: People → Alphabet

Assign each person to the first letter of her name.

A: Alphabet → Counting numbers

Assign each letter to the corresponding number 1–26.

S: Counting Numbers → Counting numbers

Assign each number to its square.

Which of the following compositions are defined? For those that are, describe the effect of the composite function.

a. $L \circ F$

The composite function is defined. The function assigns each person to his favorite animal and then to the number of legs the animal has.

b. $N \circ L$

This composition is not defined. The function L assigns each animal to its number of legs, but function N accepts only people as inputs. The range of L is not contained in the domain of N.

c. $A \circ L$

This composition is not defined. The function L assigns each animal to its number of legs, but function A accepts only letters as inputs. The range of L is not contained in the domain of A.

d. $A \circ N$

This composite function is defined. The function assigns a person to the first letter of his name, and then to the number 1–26 that corresponds to that letter.

e. $N \circ A$

This composition is not defined. The function A assigns each letter of the alphabet to its corresponding number, but function N accepts only people as inputs. The range of A is not contained in the domain of N.

f. $F \circ L$

This composition is not defined. The function L assigns each animal to its number of legs , but function F accepts only people as inputs. The range of L is not contained in the domain of F.

g. $S \circ L \circ F$

This composition is defined. The function assigns a person to her favorite animal, then to the number of legs of that animal, and then to the square of that number.

h. $A \circ A \circ N$

This composition is not defined. The composition $A \circ N$ is defined, and outputs a number, which is not a valid input to function A. The range of $A \circ N$ is not contained within the domain of A.

3. Let $f(x) = x^2 - x$, $g(x) = 1 - x$.

a. $f \circ g$

$f(g(x)) = f(1-x) = 1 - 2x + x^2 - 1 + x = x^2 - x$

b. $g \circ f$

$g(f(x)) = g(x^2 - x) = 1 - x^2 + x = -x^2 + x - 1$

c. $g \circ g$

$g(g(x)) = g(1-x) = 1 - 1 + x = x$

d. $f \circ f$

$f(f(x)) = f(x^2 - x) = x^4 - 2x^3 + x^2 - x^2 + x = x^4 - 2x^3 + x$

e. $f(g(2))$

$f(g(2)) = f(-1) = 2$

f. $g(f(-1))$

$g(f(-1)) = g(2) = -1$

4. Let $f(x) = x^2$, $g(x) = x + 3$.

a. $g(f(5))$

$g(f(5)) = g(5^2) = g(25) = 25 + 3 = 28$

b. $f(g(5))$

$f(g(5)) = f(5+3) = f(8) = 8^2 = 64$

c. $f(g(x))$

$f(g(x)) = f(x+3) = (x+3)^2 = x^2 + 6x + 9$

d. $g(f(x))$

$g(f(x)) = g(x^2) = x^2 + 3$

e. $g\left(f(\sqrt{x+3})\right)$

$g\left(f(\sqrt{x+3})\right) = g(x+3) = x + 6$

5. **Let $f(x) = x^3$, $g(x) = \sqrt[3]{x}$.**

a. $f \circ g$

$f\big(g(x)\big) = f\big(\sqrt[3]{x}\big) = x$

b. $g \circ f$

$g\big(f(x)\big) = g(x^3) = x$

c. $f\big(g(8)\big)$

$f\big(g(8)\big) = f(2) = 8$

d. $g\big(f(2)\big)$

$g\big(f(2)\big) = g(8) = 2$

e. $f\big(g(-8)\big)$

$f\big(g(-8)\big) = f(-2) = -8$

f. $g\big(f(-2)\big)$

$g\big(f(-2)\big) = g(-8) = -2$

6. **Let $f(x) = x^2$, $g(x) = \sqrt{x} + 3$.**

a. **Show that $\big(f(x+3)\big) = |x+3| + 3$.**

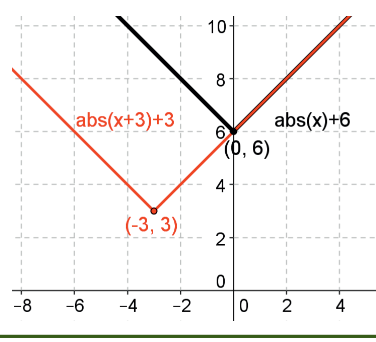
$g\big(f(x+3)\big) = g\big((x+3)^2\big) = \sqrt{(x+3)^2} + 3 = |x+3| + 3$

b. **Does $(x) = |x+3| + 3 = (x) = |x| + 6$? Graph them on the same coordinate plane.**

No, they are not equal.

For $|x+3| + 3$, ***if*** $x + 3 \geq 0$, $|x+3| + 3 = x + 6$; ***if*** $x + 3 < 0$, $|x+3| + 3 = -x$.

For $|x| + 6$, ***if*** $x \geq 0$, $|x| + 6 = x + 6$; ***if*** $x < 0$, $|x| + 6 = -x + 6$.



7. **Given the chart below, find the following:**

	-6	0	2	4
$f(x)$	4	-6	0	2
$g(x)$	2	4	-6	0
$h(x)$	0	2	4	-6
$k(x)$	1	4	0	3

a. $f(g(0))$

2

b. $g(k(2))$

4

c. $k(g(-6))$

0

d. $g(h(4))$

2

e. $g(k(4))$

$g(3)$ is not defined.

f. $(f \circ g \circ h)(2)$

-6

g. $(f \circ f \circ f)(0)$

2

h. $(f \circ g \circ h \circ g)(2)$

2

8. **Suppose a flu virus is spreading in a community. The following table shows the number of people, n, who have the virus d days after the initial outbreak. The number of people who have the virus is a function of the number of days, $n = f(d)$.**

d days	0	1	4	8	12	16	20
$n = f(d)$ number of people infected	2	4	14	32	64	50	32

There is only one pharmacy in the community. As the number of people who have the virus increases, the number of boxes of cough drops, b, sold also increases. The number of boxes of cough drops sold on a given day is a function of the number of people who have the virus, $b = g(n)$, on that day.

n number of people infected	0	2	4	9	14	20	28	32	44	48	50	60	64
$b = g(n)$ number of boxes of cough drops sold	1	5	14	16	22	30	42	58	74	86	102	124	136

a. **Find $g(f(1))$, and state the meaning of the value in the context of the flu epidemic. Include units in your answer.**

Because $f(1) = 4$ and $g(f(1)) = 14$, on day one, there were four people infected, and there were fourteen boxes of cough drops sold at the pharmacy.

b. **Fill in the chart below using the fact that $b = g(f(d))$.**

d (days)	0	1	4	8	12	16	20
b number of boxes of cough drops sold	5	14	22	58	136	102	58

c. **For each of the following expressions, interpret its meaning in the context of the problem, and if possible, give an approximation of its value.**

i. $g(f(4))$

$g(f(4)) = g(14) = 22$

On the fourth day of the outbreak, 22 boxes of cough drops were sold.

ii. $g(f(16))$

$g(f(16)) = g(50) = 102$

On the sixteenth day of the outbreak, 102 boxes of cough drops were sold.

iii. $f(g(9))$

We can compute $f(g(9)) = f(16) = 50$ but it does not make sense in the context of this problem. The output $g(9) = 16$ represents the number of boxes of cough drops sold when 9 people are infected. The output $f(16) = 50$ represents the number of people infected on day 16 of the outbreak. However, f is a function of days, not a function of the number of boxes of cough drops sold.

Lesson 17: Solving Problems by Function Composition

Student Outcomes

- Students write equations that represent functional relationships and use the equations to compose functions.
- Students analyze the domains and ranges of functions and function compositions represented by equations.
- Students solve problems by composing functions (**F-BF.A.1c**).

Lesson Notes

In the previous lesson, students explored the process of function composition in a general setting, interpreting compositions in context and determining when compositions were reasonable. This lesson focuses on composing numerical functions, including those in real-world contexts. Students represent real-world relationships with equations, use the equations to create composite functions, and apply the composite functions to problem solving in both mathematical and real-world contexts.

Classwork

Opening (3 minutes)

At the outset of the previous lesson, students were introduced to free diving, and they explored how function composition could be used to examine the relationship between the atmospheric pressure a diver experiences and the time a free diver has spent in a descent. This lesson provides students with an opportunity to represent the relationships between temperature, depth, and time spent descending using equations to address the same issue, that is, the relationship between atmospheric pressure experienced and the duration of the diver's descent. Pose the following questions after students have examined the tables provided. After a short time of individual reflection, allow students to share their ideas with a partner and then with the class.

Scaffolding:

Break down the question into parts for struggling students. For instance, ask them, "How could we represent the relationship between the time spent descending and depth?" "How could we represent the relationship between the ocean depth of the diver and atmospheric pressure?" "How could we use these representations to define the relationship between the time a diver has spent descending and the atmospheric pressure experienced by the diver?"

Depth of Free Diver During Descent

s seconds of descent	20	40	60	80	100	120	140	160	180
d depth in meters of diver	14	28	42	56	70	84	98	112	126

Atmospheric Pressure and Ocean Depth

d depth in meters of diver	10	20	30	40	50	60	70	80	90
p pressure in atmosphere on diver	2	3	4	5	6	7	8	9	10

- What patterns do you notice in the relationship between a diver's time descending and her depth and between a diver's depth and the pressure applied to the diver? How could you use this information to model the relationship between the time a diver has spent descending and the atmospheric pressure experienced by the diver?
    - *We could write a linear function representing the relationship between time and depth and another linear function relating depth and pressure and compose them to relate time and pressure.*

Discussion (8 minutes): Functions Represented with Equations

- In the previous lesson, we discussed functions generally, analyzing relationships between sets that were not always numerical. Let's explore numerical functions represented by formulas. For any function, we can define the functional relationship as $f\colon X \to Y$, and the relationship between each input and its corresponding output can be represented as $x \to f(x)$.

MP.4

- How could we use function notation to represent the following relationship? The function f takes a diver's time, in seconds, spent in descent, t, and multiplies it by 0.7 to produce the diver's depth in meters.
    - $f\colon \mathbb{R} \to \mathbb{R}$
      $t \to 0.7t$
- And how could we write an equation that represents the relationship between t and $f(t)$?
    - $f(t) = 0.7t$
- What would a reasonable domain and range be for the function $f(t) = 0.7t$?
    - *Both the domain and range represent nonnegative real numbers, and there is some upper limit given the constraints on the time and depth of a diver's descent.*
- What type of function would represent the relationship between the diver's depth and pressure applied to the diver? Explain.
    - *A linear function would best represent the relationship because the rate of change in the pressure based on a unit increase in depth is constant.*
- And how could we determine a linear equation to represent the relationship between depth and pressure?
    - *Determine the rate of change in atmospheres/meter and extrapolate the y-intercept from the table.*
- What are the rate of change and y-intercept for the equation?
    - *The rate of change is 0.1 atmosphere (atm)/meter (m), and the y-intercept is 1 atm.*
- So, what is the equation representing the relationship between depth and pressure?
    - $p(d) = 0.1d + 1$
- What would a reasonable domain and range be for the function $p(d) = 0.1d + 1$?
    - *Both the domain and range represent nonnegative real numbers, and there is some upper limit given the constraints on the depth and pressure applied.*
- How could we use our equations relating time and depth and relating depth and pressure to determine the pressure on a diver 100 seconds (sec.) into the descent?
    - *Substitute $s = 100$ into the equation $f(s) = 0.7s$ to find the diver's depth, and substitute this depth into the equation $p(d) = 0.1d + 1$ to find the pressure applied to the diver.*

- What is the resulting pressure? Explain.
  - $f(100) = 0.7(100) = 70$
    $p(70) = 0.1(70) + 1 = 8$
- Represent this process as a function composition.
  - $p\big(f(100)\big) = 8$
- Interpret this result in context.
  - *The pressure on a diver at* 100 sec. *of descent is* 8 atm.
- And how could we find an equation that represents the relationship between time and pressure?
  - *Compose the function equations for time and depth and for depth and pressure.*
- How could we compose the function equations?
  - *Evaluate the function* $p(d) = 0.1d + 1$ *for the input* $f(s) = 0.7s$.
- And why is it possible for us to use $f(s)$ as our input?
  - *It represents the diver's depth, which is equivalent to* d.
- What equation do we get when we compose the functions?
  - $p\big(f(s)\big) = p(0.7s) = 0.1(0.7s) + 1 = 0.07s + 1$
- Using this equation, what is the pressure felt by a diver 80 sec. into a descent?
  - $p\big(f(80)\big) = 0.07(80) + 1 = 6.6$

MP.2

- Explain the result in context.
  - *There are* 6.6 atm *of pressure applied to a diver* 80 sec. *into the descent.*

Example 1 (5 minutes)

This example addresses numerical functions represented with equations. Students determine the domain and range of functions given the equations. They also compose numerical functions and determine the domain and range of the compositions. This prepares them to analyze real-world relationships that can be represented using functions and compositions of functions. Have students complete parts (a) and (b) in pairs or small groups. Review parts (a) and (b), and then complete part (c) with a teacher-led discussion. Students should then complete part (d) in pairs or small groups and discuss answers in a whole-class setting. Note: It is acceptable, but not necessary, for students to represent the domain and range of the functions using interval notation.

- What is the relationship between the inputs and outputs in part (a)?
  - *Each real number* x *is squared to produce the corresponding output.*
- This function is described as a rule between sets of real numbers. This means that to determine the domain, we must consider all values of x that produce a real number output. What real numbers are included in the domain, and how can you tell?
  - *All real numbers are included because squaring a real number is an example of multiplying two real numbers, and the set of real numbers is closed under multiplication.*
- The range represents the subset of real numbers produced by applying the function rule to the domain. What type of real numbers result when a real number is squared?
  - *Nonnegative real numbers*

- What implications does this have on the range?
  - *The range can only contain elements in the real numbers that are nonnegative.*
- Why is the domain for the function in part (b) different from that in part (a)?
  - *In part (a), any real number could produce an output that is a real number. In the function in part (b), any x-value less than 2 produces an output of a nonreal complex number.*
- And why is this a problem?
  - *The sets X and Y are defined as being real number sets, which do not contain nonreal complex numbers.*
- We have seen how this impacts the domain. What effect does it have on the range?
  - *For all inputs in the domain, the output is the principal square root of a nonnegative real number. This indicates that the range only contains nonnegative real numbers.*
- How do we perform the composition $f\big(g(x)\big)$ in part (c)?
  - *Apply the function g to the inputs x, and then apply the function f to the outputs $g(x)$.*
- If $f\big(g(x)\big) = |x-2|$, and the expression $|x-2|$ produces real number outputs for all real number inputs, why is the domain of $f\big(g(x)\big)$ not all real numbers?
  - *If we apply the function rule g to the set of all real numbers, the only values that produce real number outputs are those that are greater than or equal to 2. Therefore, we cannot include values in the domain that are less than 2 because the function g would produce an output that is not a real number, and the composition could not be performed.*
- And what are the restrictions on the range of the function composition? Explain.
  - *The function g produces only nonnegative outputs, which become the inputs when function rule f is applied. Since f has the effect of squaring the input values, the output of the composite function is effectively the squared values of nonnegative real numbers, which are also nonnegative real numbers.*
- Why are the domain and range not the same for the compositions $f\big(g(x)\big)$ and $g\big(f(x)\big)$?
  - *Function composition is not commutative, so the order in which the function rules are applied affect the composite function that results.*

Example 1

Find the domain and range for the following functions:

a. $f: \mathbb{R} \to \mathbb{R}$ **given by** $f(x) = x^2$

Domain: All real numbers

Range: All real numbers greater than or equal to 0

b. $g: \mathbb{R} \to \mathbb{R}$ **given by** $g(x) = \sqrt{x-2}$

Domain: Since $x - 2 \geq 0$, $x \geq 2$

Range: $\sqrt{x-2} \geq 0$, ***so*** $g(x) \geq 0$

c. $f(g(x))$

$f(g(x)) = f\big(\sqrt{x-2}\big) = \big(\sqrt{x-2}\big)^2 = |x-2|$

Domain: $x - 2 \geq 0$, ***so*** $x \geq 2$

Range: $|x-2| \geq 0$, ***so*** $f\big(g(x)\big) \geq 0$

Scaffolding:

- Struggling students could graph the functions and their compositions to help them visualize restrictions on the domain and range.
- Advanced students could work together to draw general conclusions about the domains and ranges of function compositions compared to the domains and ranges of the individual functions being composed (e.g., if x is a restricted input value for f or g, it is also a restricted input for the composite function).

d. $g(f(x)) = g(x^2) = \sqrt{x^2 - 2}$

Domain: $x^2 - 2 \geq 0$, *so* $x \leq -\sqrt{2}$ *or* $x \geq \sqrt{2}$

Range: $\sqrt{x^2 - 2} \geq 0$, *so* $g(f(x)) \geq 0$

Exercise 1 (5 minutes)

Students should complete this problem in pairs. They should do the work independently and then verify their responses with their partners. After a few minutes, a few selected volunteers could display their work and share their solving process.

Exercise 1

1. **Find the domain and range for the following functions:**

   a. $f: \mathbb{R} \to \mathbb{R}$ **given by** $f(x) = x + 2$

   Domain: All real numbers

   Range: All real numbers

   b. $g: \mathbb{R} \to \mathbb{R}$ **given by** $g(x) = \sqrt{x - 1}$

   Domain: $x \geq 1$

   Range: $\sqrt{x - 1} \geq 0$, *so* $g(x) \geq 0$

   c. $f(g(x))$

   $f(g(x)) = f(\sqrt{x - 1}) = \sqrt{x - 1} + 2$

   Domain: $x - 1 \geq 0$, *so* $x \geq 1$

   Range: Since $\sqrt{x - 1} \geq 0$, $\sqrt{x - 1} + 2 \geq 2$, *so* $f(g(x)) \geq 2$

   d. $g(f(x))$

   $g(f(x)) = g(x + 2) = \sqrt{(x + 2) - 1} = \sqrt{x + 1}$

   Domain: $\sqrt{x + 1} \geq 0$, *so* $x \geq -1$
   Range: $\sqrt{x + 1} \geq 0$, *so* $g(f(x)) \geq 0$

Example 2 (8 minutes)

This example builds on Example 1 by having students represent real-world relationships using composite functions. Students represent the relationship between variables using equations, which they have done in previous courses. They then apply their understanding of composing functions to write an equation that relates two variables of interest (**F-BF.A.1**). Students then apply the equation to solve a problem. This problem should be completed as part of a teacher-led discussion.

- What relationship is of interest to the individuals in the nonprofit organization?
  - *The relationship of interest is between the energy generated by wind power and the energy used by electric cars.*

- What type of relationship is described between the number of turbines operating and the amount of energy they generate daily?
  - *It is a directly proportional relationship.*
- So, what is the format of the function?
  - $f(x) = kx$*, where* k *represents the constant of proportionality,* $16{,}400$
- Does it matter whether the function is written as $E(t)$ versus $f(x)$?
  - *No. It is just important to remember what the variables represent; for example,* $E(t)$ *represents energy produced in kilowatt-hours (kWh), and* t *represents the number of* 2.5–3 *megawatt (*MW*) turbines operating on the wind farm daily.*
- And how can we interpret the function we wrote in part (a)?
  - *The energy produced, in kilowatt-hours, each day by* t 2.5–3 *MW wind turbines is the product of* $16{,}400$ *and* t*.*
- What relationship is represented by our function equation in part (b)?
  - *The relationship between the energy in kilowatt-hours expended by an average electric car and miles driven by the car*
- Good. We can see that this relationship is also directly proportional. How was the constant of proportionality found?
  - *By dividing the miles driven by energy expended and rounding to an appropriate number of significant digits*
- And how can we interpret this function equation in context?
  - *An average electric car uses* 1 *kilowatt-hour of energy for every* 2.9 *miles it is driven.*
- Now how can we use the equations we have written to relate the energy generated by the turbines to the miles driven by an average electric car?
  - *Compose the functions: Apply the miles to the energy generated by the number of turbines.*
- Now our composite function equation is in the form of a directly proportional relationship. Why are the domain and range of the composite function not all real numbers?
  - *The domain represents the number of turbines, which can only be a whole number value; the range is the product of a whole number constant and a whole number input, which is also a whole number.*
- Realistically, do the domain and range for the composite function really include all whole numbers?
  - *There is some reasonable upper limit for the number of turbines that is determined by cost, available space, etc.*
- How might individuals from the nonprofit group apply the composite function in their research?
  - *Individuals could apply the composite function to their research to determine whether the energy offset by the turbines outweighs the cost of building them and maintaining them.*

Example 2

According to the Global Wind Energy Council, a wind turbine can generate about $16,400$ kWh of power each day. According to the Alternative Fuels Data Center, an average electric car can travel approximately 100 miles on 34 kWh of energy. An environmental nonprofit organization is interested in analyzing how wind power could offset the energy use of electric vehicles.

MP.4

a. **Write a function that represents the relationship between the number of wind turbines operating in a wind farm and the amount of energy they generate per day (in kilowatt-hours). Define the input and output.**

$E(t) = 16400t$
t: number of turbines operating daily
$E(t)$: energy (in kilowatt-hours) produced daily by the turbines

b. **Write a function that represents the relationship between the energy expended by an electric car (in kilowatt-hours) and the number of miles driven.**

The relationship between miles driven and energy used is directly proportional, so $m = kE$.

Since the car drives 100 miles using $34\ kWh$, $100 = k(34)$ and $k \approx 2.9$.

$m(E) = 2.9E$

E: amount of energy expended by an average electric car (in kilowatt-hours)
$m(E)$: miles driven by the electric car

c. **Write a function that could be used to determine the number of miles that an electric car could drive based on the number of wind turbines operating daily at a wind farm. Interpret this function in context.**

$m\big(E(t)\big) = m(16400t) = 2.9(16400t) = 47560t$

For every turbine operating in a wind farm daily, an average electric car can drive $47,560$ miles.

d. **Determine an appropriate domain and range for part (c). Explain why your domain and range are reasonable in this context.**

Domain: Whole numbers—The domain represents the number of turbines, which can only be represented with whole numbers.

Range: Whole numbers—Given the function $m\big(E(t)\big) = 47560t$, the outputs are found by multiplying a whole number by an input that is a whole number, which always produces a whole number. The range represents miles driven, and these values cannot be negative.

e. **How many miles of driving could be generated daily by 20 wind turbines in a day?**

$m\big(E(20)\big) = 20 \times 47560 = 951\,200$. *$951,200$ miles of driving can be generated.*

Exercises 2–3 (8 minutes)

Students could complete these exercises in small groups. Each group is assigned to complete one of the exercises. After a few minutes, the groups that completed the same exercise prepare a brief presentation of their solutions, which they could share with the rest of the students.

Exercises 2–3

2. A product safety commission is studying the effect of rapid temperature changes on the equipment of skydivers as they descend. The commission has collected data on a typical skydiver during the part of the dive when she has reached terminal velocity (maximum speed) to the time the parachute is released. They know that the terminal velocity of a diver is approximately $56\ \mathrm{m/s}$ and that, given the altitude of skydivers at terminal velocity, the temperature decreases at an average rate of $6.4\,\frac{°\mathrm{C}}{\mathrm{km}}$.

a. Write a function that represents the altitude of a skydiver experiencing terminal velocity if she reaches this speed at a height of $3{,}000\ \mathrm{m}$.

$h(s) = 3000 - 56t$

s: number of seconds spent descending at terminal velocity

$h(s)$: altitude of the skydiver (in meters)

b. Write a function that represents the relationship between the altitude of the skydiver and the temperature if the temperature at $3{,}000\ \mathrm{m}$ is $5.8°\mathrm{C}$.

MP.4

$t(h) - t(h_1) = m(h - h_1)$

$\left(h_1, t(h_1)\right) = (3000, 5.8)$

$m = -6.4\frac{°\mathrm{C}}{\mathrm{km}} = -0.0064\frac{°\mathrm{C}}{\mathrm{m}}$

$t(h) - 5.8 = -0.0064(h - 3000)$

$t(h) - 5.8 = -0.0064h + 19.2$

$t(h) = -0.0064h + 25$

h: altitude of the skydiver (in meters)

$t(h)$: temperature corresponding to the altitude of the skydiver

c. Write a function that could be used to determine the temperature, in degrees Celsius, of the air surrounding a skydiver based on the time she has spent descending at terminal velocity. Interpret the equation in context.

MP.2

$t(h(s)) = t(3000 - 56t) = -0.0064(3000 - 56t) + 25 \approx 5.8 + 0.36t$

A skydiver begins the portion of her dive at terminal velocity experiencing an air temperature of approximately $5.8°\mathrm{C}$, and the temperature increases by approximately $0.36°\mathrm{C}$ for each second of descent until she deploys the parachute.

d. Determine an appropriate domain and range for part (c).

Domain: Nonnegative real numbers

Range: Real numbers greater than or equal to 5.8

Scaffolding:

- Struggling students could be provided with the formulas for the volume of a sphere and the density of an object relative to its volume and mass.
- Advanced students could complete Exercises 2 and 3 by composing the functions without first being prompted to write the functions in parts (a) and (b).

e. **How long would it take a skydiver to reach an altitude where the temperature is $8°\text{C}$?**

For a temperature of $8°\text{C}$, $8 = 5.8 + 0.36t$
$0.36t = 2.2$, *so* $t \approx 6.1$ sec.

3. **A department store manager is planning to move some cement spheres that have served as traffic barriers for the front of her store. She is trying to determine the relationship between the mass of the spheres and their diameter in meters. She knows that the density of the cement is approximately $2,500\ \text{kg/m}^3$.**

MP.4

a. **Write a function that represents the relationship between the volume of a sphere and its diameter. Explain how you determined the equation.**

$$V(d) = \frac{1}{6}\pi d^3$$

The volume of a sphere is equal to

$$\frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = \frac{1}{6}\pi d^3$$

b. **Write a function that represents the relationship between the mass and the volume of the sphere. Explain how you determined the function.**

$m(V) = 2500V$

$\text{Mass} = \text{density} \times \text{volume} = 2500 \times V$

MP.2

c. **Write a function that could be used to determine the mass of one of the cement spheres based on its diameter. Interpret the equation in context.**

$$m(V(d)) = m\left(\frac{1}{6}\pi d^3\right) = 2500\left(\frac{1}{6}\pi d^3\right) = \frac{2500}{6}\pi d^3$$

The numerical value of the mass of one of the cement spheres is equal to approximately $1,300$ *times the value of the cubed diameter (measured in meters).*

d. **Determine an appropriate domain and range for part (c).**

Domain: Nonnegative real numbers

Range: Nonnegative real numbers

e. **What is the approximate mass of a sphere with a diameter of 0.9 m?**

$m(V(0.9)) = \frac{2500}{6}\pi(0.9)^3 \approx 950$. The mass of the sphere is approximately 950 kg.

Closing (3 minutes)

Have students explain how function composition could be applied to represent the relationship between a person's income and interest earned in a savings account if there is a functional relationship between the individual's income and the amount deposited into the savings account. Students should provide a hypothetical example, which they could share with a partner.

Answers will vary. An example of an acceptable response is shown. Let's say that an individual puts a constant percent of her income, p, into a savings account. The relationship between income and the amount placed into the savings account could be defined as $s(d) = pd$, where $s(d)$ is the amount, in dollars, placed in the savings account, and d represents income in dollars. The function $I(s) = rs$ could represent the relationship between interest dollars and the amount in a savings account, where $I(s)$ represents savings interest in dollars, r is the percent interest rate, and s is the amount, in dollars, placed into savings. The relationship between income and savings interest, then, would be represented by the composite function $I\big(s(d)\big)$.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 17: Solving Problems by Function Composition

Exit Ticket

Timmy wants to install a wooden floor in a square room. The cost to install the floor is $\$24$ per 4 square feet.

a. Write a function to find the area of the room as a function of its length.

b. Write a function for the cost to install the floor as a function of its area.

c. Write a function to find the total cost to install the floor.

d. Show how the function in part (c) is the result of a composition of two functions.

e. How much does it cost to install a wood floor in a square room with a side length of 10 feet?

Exit Ticket Sample Solutions

Timmy wants to install a wooden floor in a square room. The cost to install the floor is $\$24$ per 4 square feet.

a. **Write a function to find the area of the room.**

$A(x) = x^2$

x: *a side of a square*

$A(x)$: *the area of the square room*

b. **Write a function for the installation cost per square foot.**

$C(A) = 6A$

A: *the area of the floor*

$C(A)$: *installation cost per square foot*

c. **Write the function to find the total cost to install the floor.**

$C(A(x)) = 6x^2$

d. **Show how the function in part (c) is the result of a composition of two functions.**

$A(x) = x^2$

$C(A(x)) = C(x^2) = 6(x^2) = 6x^2$

e. **How much does it cost to install a wood floor in a square room with a side of 10 feet?**

$C(A(10)) = 6(10)^2 = 600$. *The cost is* $\$600.00$.

Problem Set Sample Solutions

Students may use graphing calculators to determine domain and range for Problem 1.

1. **Find the domain and range of the following functions:**

a. $f: \mathbb{R} \to \mathbb{R}$ **by** $f(x) = -x^2 + 2$

Domain: x: *all real numbers*

Range: $f(x) \leq 2$

b. $f: \mathbb{R} \to \mathbb{R}$ **by** $f(x) = \frac{1}{x+1}$

Domain: $x \neq -1$

Range: $f(x) \neq 0$

c. $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \sqrt{4 - x}$

Domain: $x \leq 4$

Range: $f(x) \geq 0$

d. $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$

Domain: x: *all real numbers*

Range: $f(x) \geq 0$

e. $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 2^{x+2}$

Domain: x: *all real numbers*

Range: $f(x) > 0$

2. Given $f: \mathbb{R} \to \mathbb{R}$ and: $\mathbb{R} \to \mathbb{R}$, for the following, find $f\big(g(x)\big)$ and $g\big(f(x)\big)$, and state the domain.

a. $f(x) = x^2 - x, \;\; g(x) = x - 1$

$f\big(g(x)\big) = x^2 - 3x + 2,$ *Domain:* x: *all real numbers*

$g\big(f(x)\big) = x^2 - x - 1,$ *Domain:* x: *all real numbers*

b. $f(x) = x^2 - x, \;\; g(x) = \sqrt{x - 2}$

$f\big(g(x)\big) = x - 2 - \sqrt{x - 2},$ *Domain:* $x \geq 2$

$g\big(f(x)\big) = \sqrt{x^2 - x - 2},$ *Domain:* $x \leq -1$ *or* $x \geq 2$

c. $f(x) = x^2, \;\; g(x) = \frac{1}{x-1}$

$f\big(g(x)\big) = \frac{1}{(x-1)^2},$ *Domain:* $x \neq 1$

$g\big(f(x)\big) = \frac{1}{x^2-1},$ *Domain:* $x \neq \pm 1$

d. $f(x) = \frac{1}{x+2}, \;\; g(x) = \frac{1}{x-1}$

$f\big(g(x)\big) = \frac{x-1}{2x-1},$ *Domain:* $x \neq 1$ *and* $x \neq \frac{1}{2}$

$g\big(f(x)\big) = \frac{x+2}{-x-1},$ *Domain:* $x \neq -2$ *and* $x \neq -1$

e. $f(x) = x - 1, \;\; g(x) = \log_2(x + 3)$

$f\big(g(x)\big) = \log_2(x + 3) - 1,$ *Domain:* $x > -3$

$g\big(f(x)\big) = \log_2(x + 2),$ *Domain:* $x > -2$

3. A company has developed a new highly efficient solar panel. Each panel can produce 0.75 MW of electricity each day. According to the Los Angeles power authority, all the traffic lights in the city draw 0.5 MW of power per day.

   a. Write a function that represents the relationship between the number of solar panels installed and the amount of energy generated per day (in MWh). Define the input and output.

   $E(n) = 0.75n$

   n is the number of panels operating in one day.

   E is the total energy generated by the n panels.

   b. Write a function that represents the relationship between the number of days and the energy in megawatts consumed by the traffic lights. (How many days can one megawatt provide?)

   $D(E) = \frac{1}{0.5}E = 2E$

   E: the energy in megawatts

   $D(E)$: the number of days per megawatt

   c. *Write a function that could be used to determine the number of days that the traffic lights stay on based on the number of panels installed.*

   $D\big(E(n)\big) = 2E(n) = 2 \times 0.75n = 1.5n$

   d. Determine an appropriate domain and range for part (c).

   Domain: whole numbers

   Range: whole numbers (whole number multiples of 1.5)

   e. How many days can 20 panels power all the lights?

   $D\big(E(20)\big) = 1.5 \times 20 = 30$. *It takes 30 days.*

4. A water delivery person is trying to determine the relationship between the mass of the cylindrical containers he delivers and their diameter in centimeters. The density of the bottles is $1\ \mathrm{g/cm^3}$. The height of each bottle is approximately 60 cm.

   a. Write a function that represents the relationship between the volume of the cylinder and its diameter.

   $V(d) = 15\pi d^2$

   b. Write a function that represents the relationship between the mass and volume of the cylinder.

   $m(V) = 1V$

   c. Write a function that could be used to determine the mass of one cylinder based on its diameter. Interpret the equation in context.

   $D\big(V(d)\big) = 1 \times 15\pi d^2 = 15\pi d^2$

   The numerical value of the mass of one of the cylindrical water containers is equal to approximately 47.1 times the value of the squared diameter.

d. Determine an appropriate domain and range for part (c).

Domain: nonnegative real numbers

Range: nonnegative real numbers

e. What is the approximate mass of a cylinder with a diameter of 30 cm?

$\boldsymbol{D\big(V(60)\big) = 15\pi(30)^2 \approx 42411\ g \approx 42.4.}$

The mass is approximately 42.4 kg.

5. A gold mining company is mining gold in Northern California. Each mining cart carries an average 500 kg of dirt and rocks that contain gold from the tunnel. For each 2 metric tons of material (dirt and rocks), the company can extract an average of 10 g of gold. The average wholesale gold price is $\$20$/g.

a. Write a function that represents the relationship between the mass of the material mined in metric tons and the number of carts. Define the input and output.

$\boldsymbol{V(n) = 0.5n}$

n is the number of carts.

$V(n)$ is the total mass of dirt and rock carried out by the n carts in metric tons.

b. Write a function that represents the relationship between the amount of gold and the materials. Define the input and output.

$\boldsymbol{G(V) = 0.000005V}$

V: the amount of material in metric tons

$G(V)$: the mass of gold in metric tons

c. Write a function that could be used to determine the mass of gold in metric tons as a function of the number of carts coming out from the mine.

$\boldsymbol{G\big(V(n)\big) = 0.000\,002\,5n}$

d. Determine an appropriate domain and range for part (c).

Domain: whole numbers

Range: positive real numbers

e. Write a function that could be used to determine the amount of money the gold is worth in dollars and the amount of gold extracted in metric tons.

$\boldsymbol{C\big(V(n)\big) = 20\,000\,000 \times 0.000\,002\,5n = 50n}$

f. How much gold can $40,000$ carts of material produce?

$\boldsymbol{G\big(V(40000)\big) = 0.1}$ *metric ton*

g. How much, in dollars, can $40,000$ carts of material produce?

$\boldsymbol{C\big(V(40000)\big) = 50 \times 40000 = 2\,000\,000}$ *The cost is $\$2\,000\,000$.*

6. **Bob operates hot air balloon rides for tourists at the beach. The hot air balloon rises, on average, at 100 feet per minute. At sea level, the atmospheric pressure, measured in inches of mercury (inHg), is 29.9 inHg. Using a barometric meter, Bob notices that the pressure decreases by 0.5 inHg for each 500 feet the balloon rises.**

   a. **Write a function that represents the relationship between the height of the hot air balloon and the time spent to reach that height.**

   $H(t) = 100t$

   t is the number of minutes.

   $H(t)$ is the height of the hot air balloon at t time.

   b. **Write a function that represents the relationship between the height of the hot air balloon and the atmospheric pressure being applied to the balloon.**

   $P(H) = 29.9 - \frac{0.5H}{500}$

   H: the height of the hot air balloon

   $P(H)$: the reading on the barometer at the height of the hot air balloon

   c. **Write a function that could be used to determine the pressure on the hot air balloon based on the time it spends rising.**

   $P\big(H(t)\big) = 29.9 - \frac{0.5H(t)}{500} = 29.9 - 0.1t$

   d. **Determine an appropriate domain and range for part (c).**

   Domain: nonnegative real numbers

   Range: nonnegative real numbers

   e. **What is the reading on the barometer 10 minutes after the hot air balloon has left the ground?**

   $P\big(H(10)\big) = 29.9 - 0.1t = 28.9$ *The reading is* $.9$ inHg.

Topic C

Inverse Functions

F-BF.B.4b, F-BF.B.4c, F-BF.B.4d, F-BF.B.5

Focus Standards:	F-BF.B.4	Find inverse functions. b. (+) Verify by composition that one function is the inverse of another. c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse. d. (+) Produce an invertible function from a non-invertible function by restricting the domain.
	F-BF.B.5	(+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.
Instructional Days:	4	
Lesson 18:	Inverse Functions (P)[1]	
Lesson 19:	Restricting the Domain (P)	
Lesson 20:	Inverses of Logarithmic and Exponential Functions (P)	
Lesson 21:	Logarithmic and Exponential Problem Solving (E)	

The inverse of a function and the associated properties of inverses were first introduced in Algebra II Module 3. Students solved for inverse functions by interchanging x and y and discovered that the graphs of inverse functions result from reflecting the original graph over the line $y = x$. Students studied the relationship between exponential and logarithmic functions graphically and in problem solving. Lesson 18 presents inverses in a slightly different context as students understand that when a function and its inverse are composed, they undo each other (**F-BF.B.4b**). Students then create the inverse of a function algebraically.

In Lesson 19, students work with functions and their inverses represented numerically, graphically, and algebraically. This lesson builds on student understanding of inverses as they explore the inverse of a function in the context of tables and graphs (**F-BF.B.4c**). They also consider that not every function has an inverse that is also a function and learn how to restrict the domain of a function to produce an invertible function (**F-BF.B.4d**), setting the stage for the definition of the inverse trigonometric functions in Module 4. The lesson defines the adjective invertible (a function is invertible if its inverse is also a function) as it applies to functions and provides practice for students to verify by composition that two functions are inverses (**F-BF.B.4b**).

[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Lessons 20 and 21 focus on logarithmic and exponential functions. Students have worked with exponential functions since Algebra I, and logarithmic functions were introduced and studied extensively in Algebra II. In fact, the inverse of a function was first defined in Algebra II because students needed to be able to solve equations of the form $a \cdot b^{cx} = d$ and needed to understand that the solution to this type of equation is a logarithm. In Lesson 20, students review the inverse relationship between exponents and logarithms (**F-BF.B.5**) and use composition to verify that a logarithmic function and an exponential function are inverses (**F-BF.B.4d**). Lesson 21 wraps up Topic C and Module 3 as students solve modeling problems using exponential and logarithmic functions (**F-BF.B.5**).

Mathematical practices MP.2 and MP.4 are highlighted as students consider a simple linear model of straight-line depreciation in a business application, relating parameters of the function to the context. Students also study carbon dating techniques used to determine the age of woolly mammoth remains and write exponential and logarithmic models. Students use graphing calculators and graphing software to explore the relationships between functions and their inverses and to problem solve (MP.5).

Lesson 18: Inverse Functions

Student Outcomes

- Students read the inverse values of a function from a table and graph. They create the inverse of a function by solving an equation of the form $f(x) = y$. They understand the definition of the inverse of a function and properties that relate a function to its inverse.

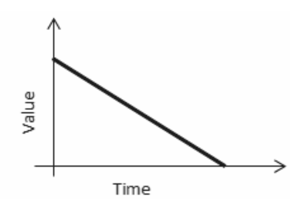
Lesson Notes

This lesson reintroduces the inverse of a function and begins to address the additional standards that are part of **F-BF.4**. Specifically in this lesson, students create the inverse of a function of the form $f(x) = y$ (**F-BF.B.4a**) and read values of an inverse function from a graph or table (**F-BF.B.4c**). Students consider a simple linear situation in the context of straight-line depreciation of business equipment (MP.4 and MP.2). They work with inverses presented graphically and in tables (**F-BF.B.4b**).

The inverse of a function and the associated properties of inverses were first introduced in Algebra II Module 3 Lessons 18 and 19. The goals of this lesson are to present inverses in a slightly different context: to help students understand that when a function and its inverse are composed, they undo each other, and to review a process for creating the inverse of a function algebraically.

Scaffolding:

- Add unfamiliar words such as *asset* and *depreciation* to a word wall. Ask students to brainstorm assets that businesses might have (e.g., computers, equipment, buildings, and vehicles) and construct a simple diagram illustrating straight-line depreciation like the one shown below.



- Depreciation can be represented numerically in a table.

Time	0	1	2
Value	$1{,}000$	900	800

- Ask students to explain the meaning of ordered pairs from the table, such as $(0, 1000)$ or $(1, 900)$.

Classwork

Opening (3 minutes)

Ask students to read the opening paragraph silently and highlight any terms that seem unfamiliar to them. Lead a short discussion to access students' prior knowledge about depreciation.

Businesses must track the value of their assets over time. When a business buys equipment, the value of the equipment is reduced over time. For example, electric companies provide trucks for their workers when they go out into the field to repair electrical lines. These trucks lose value over time but are still part of the business assets. For accounting purposes, many businesses use a technique called *straight-line depreciation* to calculate the value of equipment over time.

Exercises 1–5 (7 minutes)

Have students work these exercises with a partner or in small groups. Students have worked with functions extensively in Algebra I and Algebra II and throughout this module, so these exercises should be fairly easy for them. Key words like *domain, range,* and *parameter* can be reviewed briefly as necessary. Students should be encouraged to use a calculator as needed for computation and graphing since the object of this lesson is for them to understand the concept of inverse functions, especially in real-world situations.

Exercises

Suppose ABC Electric purchases a new work truck for $\$34,500$. They estimate that the truck's value will depreciate to $\$0$ over 15 years. The table below shows the value $v(t)$ of the truck in thousands of dollars depreciated over time t in months using a straight-line depreciation method.

t	0	12	24	36	48	60	72	84	96
$v(t)$	34.5	32.2	29.9	27.6	25.3	23.0	20.7	18.4	16.1

1. **Does the function v appear to be a linear function? Explain your reasoning.**

   Yes. Each time the months increase by 12, the value decreases by 2.3.

2. **What is an appropriate domain and range for v in this situation?**

   The domain would be $[0, 180]$, which represents a time span of 15 years measured in months. The range would be $[0, 34.5]$, which represents the value of the car over the 15-year period based on the company's estimates.

3. **Write a formula for v in terms of t, the months since the truck was purchased.**

   $v(t) = -\frac{2.3}{12}t + 34.5$ ***for*** $0 \leq t \leq 180$

4. **What will the truck be worth after 30 months? 40 months? 50 months?**

   All values are rounded to the hundredths place.

$$v(30) = 28.75$$
$$v(40) = 26.83$$
$$v(50) = 24.92$$

5. **When will the truck be valued at $\$30,000$? $\$20,000$? $\$10,000$?**

   $v(t) = 30$ ***when***

$$-\frac{2.3}{12}x + 34.5 = 30$$

To solve for t, subtract both sides of the equation by 34.5, and then multiply both sides by $-\frac{12}{2.3}$.

$$-\frac{2.3}{12}t + 34.5 - 34.5 = 30 - 34.5$$
$$-\frac{2.3}{12}t = -4.5$$
$$-\frac{12}{2.3}\left(-\frac{2.3}{12}t\right) = -\frac{12}{2.3}(-4.5)$$
$$t \approx 23.48$$

After approximately 23.5 months, the truck will be worth $\$30,000$.

Similarly, solving the equations $v(t) = 20000$ and $v(t) = 10000$ will give approximate times for when the truck is worth $\$20,000$ and $\$10,000$. The truck will be worth $\$20,000$ after approximately 75.7 months, and it will be worth $\$10,000$ after approximately 127.8 months.

Discussion (3 minutes)

Lead a short discussion to debrief Exercises 1–5 before moving on to the next few problems.

MP.2

- What is the meaning of the parameters in the linear function v?
  - *The 34.5 represents the original price of the truck in thousands of dollars. The $-\frac{2.3}{12}$ is the monthly price decrease due to the straight-line depreciation.*
- How did you determine the answers to Exercise 5? Did you notice any similarities as you determined the months for the different dollar amounts?
  - *To solve the equation, you subtract 34.5 each time and then multiply by the reciprocal of $-\frac{2.3}{12}$. You always undo the operations in the reverse order that they would have been applied to the t in the equation to create the expression on the left side of the equation.*
- How could you write a formula that would give the time for any dollar amount?
  - *You could solve the equation $-\frac{2.3}{12}t + 34.5 = y$ for t, and that would give you a formula to find the time for any dollar amount.*

Exercises 6–10 (5 minutes)

Have students continue with these exercises working either with a partner or in small groups.

6. **Construct a table that shows the time of depreciation, $t(v)$, in months as a function of the value of the truck, v, in thousands of dollars.**

v	34.5	32.2	29.9	27.6	25.3	23.0	20.7	18.4	16.1
$t(v)$	0	12	24	36	48	60	72	84	96

7. **Does the function t appear to be a linear function? Explain your reasoning.**

   Yes. Each time the v values decrease by 2.3, the $t(v)$ values increase by 12.

8. **What is an appropriate domain and range for t in this situation?**

   The domain and range are the same as for the function v, except switched.

9. **Write a formula for t in terms of the value of the truck, v, since it was purchased.**

   Using the slope of the t function, $-\frac{12}{2.3}$, and a point $(34.5, 0)$, we could create the equation $t(v) = -\frac{12}{2.3}(v - 34.5) + 0$.

10. Explain how you can create the formula for t using the formula for v from Exercise 5.

Solve $v(t) = y$ for t, and then just change the variables.

$$-\frac{2.3}{12}t + 34.5 = y$$
$$-\frac{2.3}{12}t = y - 34.5$$
$$t = -\frac{12}{2.3}(y - 34.5)$$

Thus, $t(v) = -\frac{12}{2.3}(v - 34.5)$ for $0 \leq v \leq 34.5$.

Discussion (5 minutes)

The process that students used in Exercise 5 and then generalized in Exercise 10 is a direct application of **F-BF.B.4a**. The goal of this Discussion is to reactivate students' previous learning about inverse functions. If students mention the term *inverse,* then tailor the Discussion to what students recall. If the term does not come up, then this Discussion can be used to help students recall what they learned in Algebra II.

- What do you notice about the domains and ranges of the functions v and t?
  - *The domain of v is the range of t, and the range of v is the domain of t.*
- If a point (a, b) is on the graph of v, then what would be a point on the graph of t?
  - *Since the domain and range are switched between these two functions, (b, a) would be a point on the graph of t.*
- How could you prove that if (a, b) is on the graph of v, then (b, a) would be a point on the graph of t?
  - *If (a, b) is a point on the graph of v, then $v(a) = b$, which means that $-\frac{2.3}{12}a + 34.5 = b$. We need to show that $t(b) = a$. $t(b) = -\frac{12}{2.3}(b - 34.5) = -\frac{12}{2.3}\left(-\frac{2.3}{12}a + 34.5 - 34.5\right) = a$*

Tell students that the functions v and t are called *inverse functions*. The definition is provided below and is also provided in the Lesson Summary. Consider having students record this definition in their notebooks, share the definition in their own words with a neighbor, and come up with a few examples to include in their notebooks. Emphasize which exercises illustrate each part of the definition.

Scaffoldinging:

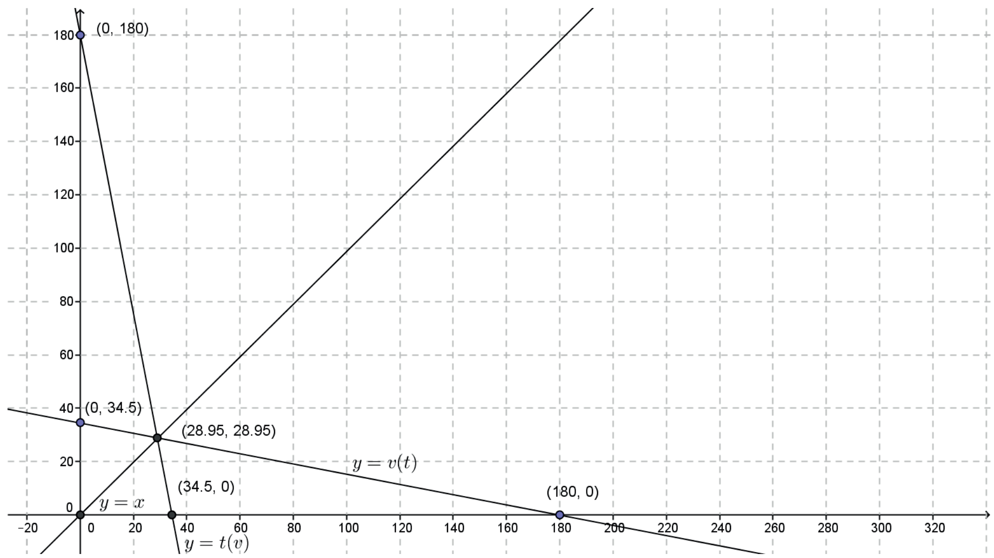
Students can create a Frayer model to have as a graphic organizer to summarize the definition and examples.

THE INVERSE OF A FUNCTION: Let f be a function with domain set X and range set Y. Then, f is *invertible* if there exists a function g with domain Y and range X such that f and g satisfy the property:
For all x in X and y in Y, $f(x) = y$ if and only if $g(y) = x$.
The function g is called the *inverse* of f and is often denoted f^{-1}.
If f and g are inverses of each other, then:
The domain of f is the same set as the range of g.
The range of f is the same set as the domain of g.

Exercises 11–13 (5 minutes)

Students can use technology to graph the functions v and t. To see the symmetry, make sure that the viewing window of the graphing calculator is set up so the scaling is equal on both axes. Most graphing calculators have a square window setting that can be used to correct the scaling when the LCD screen is rectangular.

11. Sketch a graph of the equations $y = v(t)$ and $y = t(v)$ in the Cartesian plane. How do their graphs compare?



Both graphs are decreasing, and they intersect at the point $(28.95, 28.95)$.

12. What is the meaning of the intersection point of the graphs of the two equations?

It is the point where the value of the truck (in thousands of dollars) is equal to the number of months that its value has depreciated since its purchase.

Note to teachers: There is not any significant real-world meaning to this point, but it helps to set up the fact that these functions are a reflection of one another across the line $y = x$.

13. Add the graph of $y = x$ to your work in Exercise 11. Describe the relationship between the graphs of $y = v(t)$, $y = t(v)$, and $y = x$.

The graphs of $y = v(t)$ and $y = t(v)$ are images of one another across the line $y = x$.

Debrief these exercises by reviewing how to construct the image of a point reflected across a line. The points are equidistant from the reflection line and located on a line that lies perpendicular to the reflection line. This property was a major focus of Algebra II Module 3 Lesson 18. If students are struggling with this type of transformation, review this lesson with them.

REFLECTION PROPERTY OF A FUNCTION AND ITS INVERSE FUNCTION: If two functions whose domain and range are a subset of the real numbers are inverses, then their graphs are reflections of each other across the diagonal line given by $y = x$ in the Cartesian plane.

Exercises 14–15 (10 minutes)

Exercise 14 provides students with the opportunity to work with inverses in a real-world situation again. This time, students are given a formula for a depreciation function. In Exercise 15, students work to find the inverse of an algebraic function. Depending on how much information students recall from Algebra II Module 3 Lesson 19, choose to model one of these exercises directly with the entire class.

Scaffolding:
- To students who finish early, pose the challenge to verify the inverses they found by composing them with the original functions.
- Students may need to be reminded that this property is expressed in the definition by the statement, "For all x in X and y in Y, $f(x) = y$ if and only if $g(y) = x$."
- If students struggle to verify by composition algebraically, have them practice the skill by composing the functions for individual points, for instance, $(3, -8)$ for part (a) and $(1, 250)$ for part (b).

14. ABC Electric uses this formula, $f(x) = 750 - 10x$, to depreciate computers, where f is the value of a computer and x is the number of months since its purchase.

a. Calculate $f(36)$. What is the meaning of $f(36)$?

$f(36) = 750 - 10(36) = 390$

It is the value of a computer 36 months after the date it is purchased.

b. What is the meaning of b in $f(b) = 60$? What is the value of b?

It is the number of months since its purchase date when the computer will be worth $\$60$. Solving $750 - 10b = 60$ gives a value of $b = 69$.

c. Write a formula for f^{-1}, and explain what it means in this situation.

$$f^{-1}(x) = \frac{x - 750}{-10}$$

This function determines the number of months that have passed since the computer's date of purchase given the value of the computer, x dollars.

d. When will the depreciated value of a computer be less than $\$400$?

Evaluate $f^{-1}(400)$.

$$f^{-1}(x) = \frac{400 - 750}{-10} = \frac{-350}{-10} = 35$$

The value of a computer will be less than $\$400$ after 35 months.

e. What is the meaning of c in $f^{-1}(c) = 60$? What is the value of c?

It is the value of the computer 60 months after its date of purchase.

To find the value of c, you can evaluate $f(60)$, or you can solve the equation $\frac{c - 750}{-10} = 60$ for c.

The value of c is 150. After 60 months, the value of the computer is $\$150$.

Have one or two students present their solutions to this problem, and then review the process for finding the inverse of a function algebraically, which was detailed in Module 3 Lesson 19. This review can take place either before or after this exercise is completed, depending on how much students remember about inverses of functions from Algebra II Module 3. If time permits, let them struggle a bit to work through these exercises, and then bring the class together to describe a process that works every time for finding an inverse, provided the original function has an inverse that is a function (see the Lesson Summary). All of the functions in this lesson are invertible. Non-invertible functions are discussed in the next lesson.

15. Find the inverses of the following functions:

a. $f(x) = \frac{2}{3}x - 10$

$$y = \frac{2}{3}x - 10$$
$$x = \frac{2}{3}y - 10$$
$$x + 10 = \frac{2}{3}y$$
$$y = \frac{3}{2}(x + 10)$$
$$f^{-1}(x) = \frac{3}{2}(x + 10)$$

b. $g(x) = 2(x + 4)^3$

$$y = 2(x + 4)^3$$
$$x = 2(y + 4)^3$$
$$\frac{x}{2} = (y + 4)^3$$
$$\sqrt[3]{\frac{x}{2}} = y + 4$$
$$y = \sqrt[3]{\frac{x}{2}} - 4$$
$$g^{-1}(x) = \sqrt[3]{\frac{x}{2}} - 4$$

c. $h(x) = \frac{1}{x - 2}, x \neq 2$

$$y = \frac{1}{x - 2}$$
$$x = \frac{1}{y - 2}$$
$$x(y - 2) = 1$$
$$xy - 2x = 1$$
$$xy = 1 + 2x$$
$$y = \frac{1 + 2x}{x}, x \neq 0$$
$$h^{-1}(x) = \frac{1 + 2x}{x}, x \neq 0$$

Closing (3 minutes)

Have students respond to the following questions in writing, individually or with a partner, to close this lesson. The question answers are sample student responses.

- What are two things you learned about a function and its inverse function?
  - *The domain and range are switched. The graphs are reflections of one another across the line $y = x$.*
- What is a question you still have about a function and its inverse function?
  - *Do all functions have an inverse? How do I prove that two functions are inverses?*

The Lesson Summary can be used to clarify any questions students may still have.

Lesson Summary

- **INVERTIBLE FUNCTION: Let f be a function whose domain is the set X and whose image (range) is the set Y. Then, f is *invertible* if there exists a function g with domain Y and image (range) X such that f and g satisfy the property:**

  For all x in X and y in Y, $f(x) = y$ if and only if $g(y) = x$.

  The function g is called the *inverse* of f.
- **If two functions whose domain and range are a subset of the real numbers are inverses, then their graphs are reflections of each other across the diagonal line given by $y = x$ in the Cartesian plane.**
- **If f and g are inverses of each other, then:**
  - **The domain of f is the same set as the range of g.**
  - **The range of f is the same set as the domain of g.**
- **The inverse of a function f is denoted f^{-1}.**
- **In general, to find the formula for an inverse function g of a given function f:**
  - **Write $y = f(x)$ using the formula for f.**
  - **Interchange the symbols x and y to get $x = f(y)$.**
  - **Solve the equation for y to write y as an expression in x.**
  - **Then, the formula for f^{-1} is the expression in x found in the previous step.**

Exit Ticket (4 minutes)

Name ____ Date ____________________

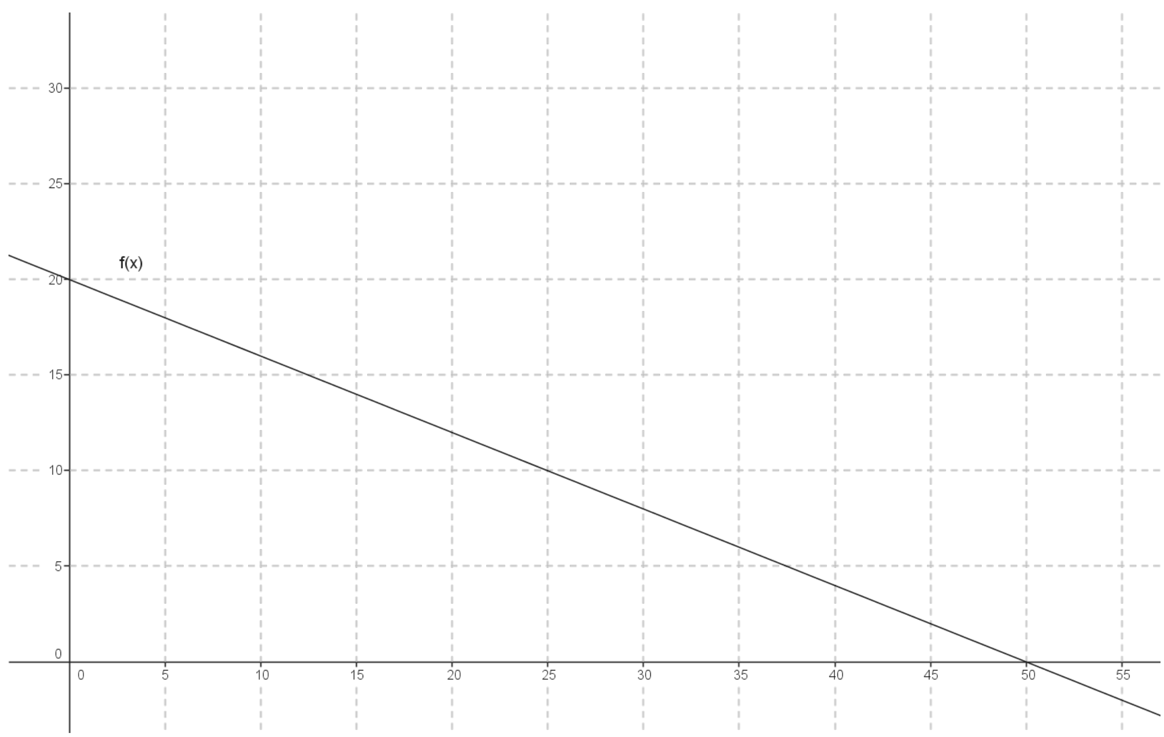
Lesson 18: Inverse Functions

Exit Ticket

The function f is described below in three different ways. For each way, express f^{-1} in the same style.

x	1	2	5	10	15	20
$f(x)$	19.6	19.2	18	16	14	12

$$f(x) = -\frac{2}{5}x + 20$$



Exit Ticket Sample Solutions

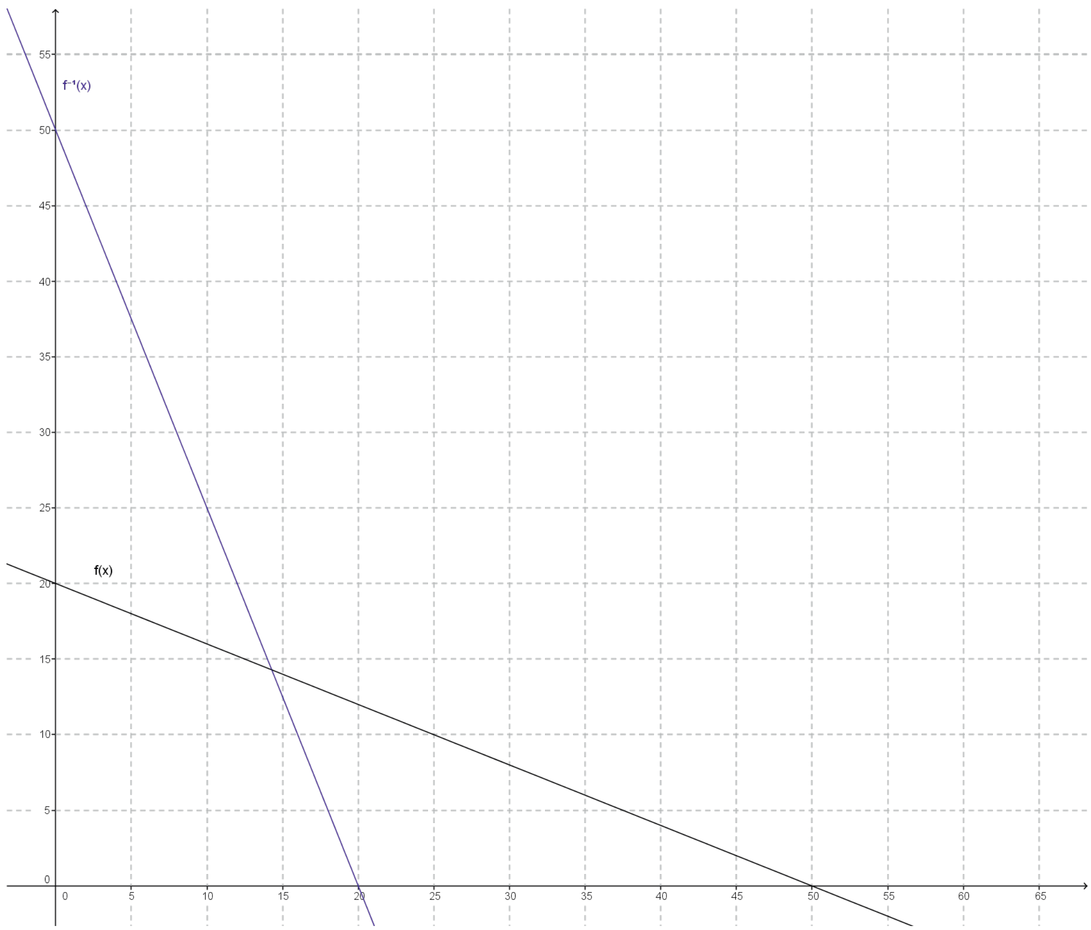
The function f is described below in three different ways. For each way, express f^{-1} in the same style.

x	1	2	5	10	15	20
$f(x)$	19.6	19.2	18	16	14	12

x	19.6	19.2	18	16	14	12
$f^{-1}(x)$	1	2	5	10	15	20

$$f(x) = -\frac{2}{5}x + 20$$

$$f^{-1}(x) = -\frac{5}{2}x + 50$$



Problem Set Sample Solutions

1. For each of the following, write the inverse of the function given.

a. $f = \{(1,3), (2,15), (3,8), (4,-2), (5,0)\}$

$f^{-1} = \{(3,1), (15,2), (8,3), (-2,4), (0,5)\}$

b. $g = \{(0,5), (2,10), (4,15), (6,20)\}$

$g^{-1} = \{(5,0), (10,2), (15,4), (20,6)\}$

c. $h = \{(1,5), (2,25), (3,125), (4,625)\}$

$h^{-1} = \{(5,1), (25,2), (125,3), (625,4)\}$

d.

x	1	2	3	4
$f(x)$	3	12	27	48

x	3	12	27	48
$f^{-1}(x)$	1	2	3	4

e.

x	-1	0	1	2
$g(x)$	3	6	12	24

x	3	6	12	24
$g^{-1}(x)$	-1	0	1	2

f.

x	1	10	100	$1{,}000$
$h(x)$	0	1	2	3

x	0	1	2	3
$h^{-1}(x)$	1	10	100	$1{,}000$

g. $y = 2x$

$y = \frac{1}{2}x$

h. $y = \frac{1}{3}x$

$y = 3x$

i. $y = x - 3$

$y = x + 3$

j. $y = -\frac{2}{3}x + 5$

$y = -\frac{3}{2}x + \frac{15}{2}$

k. $2x - 5y = 1$

$2y - 5x = 1$

l. $-3x + 7y = 14$

$-3y + 7x = 14$

m. $y = \frac{1}{3}(x - 9)^3$

$y = \sqrt[3]{3x} + 9$

n. $y = \frac{5}{3x - 4}, x \neq \frac{4}{3}$

$y = \frac{5}{3x} + \frac{4}{3}$

o. $y = 2x^7 + 1$

$y = \sqrt[7]{\frac{1}{2}x - \frac{1}{2}}$

p. $y = \sqrt[5]{x}$

$y = x^5$

q. $y = \frac{x + 1}{x - 1}, x \neq 1$

$y = \frac{x + 1}{x - 1}$

2. **For each part in Problem 1, state the domain, D, and range, R, of the inverse function.**

a. $D = \{-2, 0, 3, 8, 15\}$
$R = \{0, 1, 2, 3, 4, 5\}$

b. $D = \{5, 10, 15, 20\}$
$R = \{0, 2, 4, 6\}$

c. $D = \{5, 25, 125, 625\}$
$R = \{1, 2, 3, 4\}$

d. $D = \{3, 12, 27, 48\}$
$R = \{1, 2, 3, 4\}$

e. $D = \{3, 6, 12, 24\}$
$R = \{-1, 0, 1, 2\}$

f. $D = \{0, 1, 2, 3\}$
$R = \{1, 10, 100, 1000\}$

g. *Both domain and range are all real numbers.*

h. *Both domain and range are all real numbers.*

i. ***Both domain and range are all real numbers.***

j. ***Both domain and range are all real numbers.***

k. ***Both domain and range are all real numbers.***

l. ***Both domain and range are all real numbers.***

m. ***Both domain and range are all real numbers.***

n. ***The domain is all real numbers except $x = 0$, and the range is all real numbers except $y = \frac{4}{3}$.***

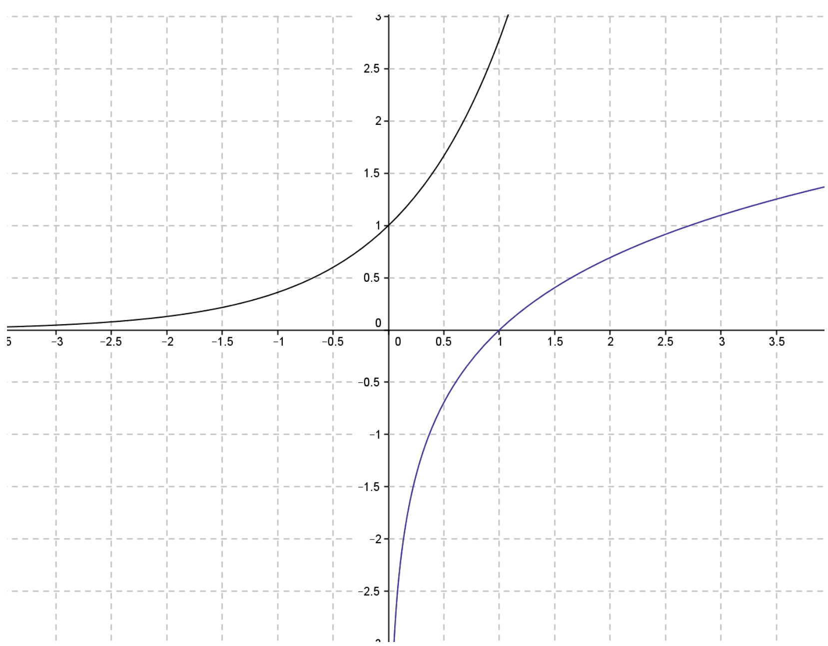
o. ***Both domain and range are all real numbers.***

p. ***Both domain and range are all real numbers.***

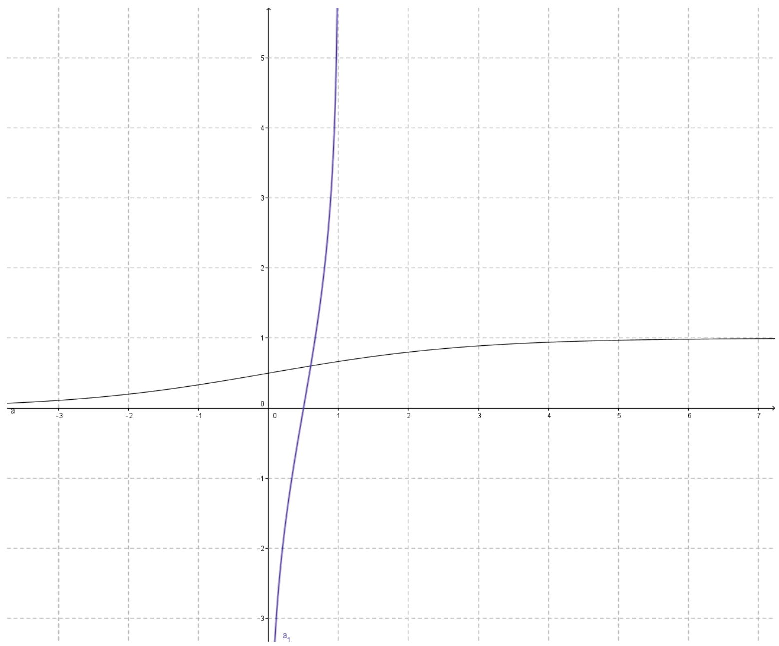
q. ***Both domain and range are all real numbers except 1.***

3. **Sketch the graph of the inverse function for each of the following functions:**

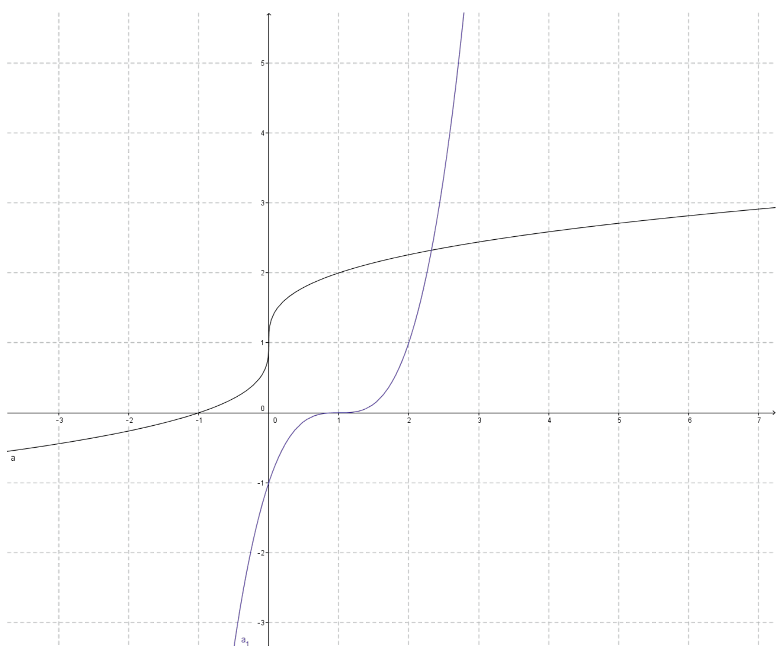
a.



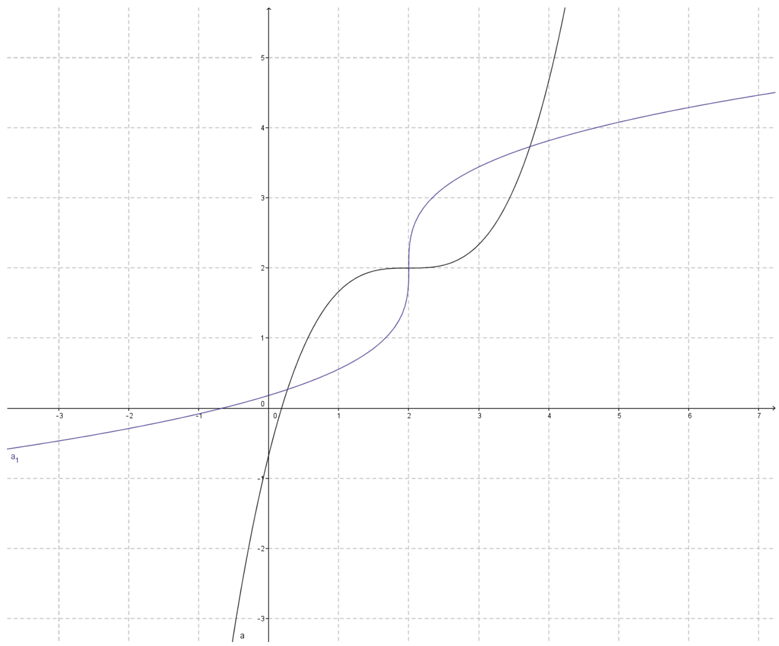
b.



c.



d.



4. Natalie thinks that the inverse of $f(x) = x - 5$ is $g(x) = 5 - x$. To justify her answer, she calculates $f(5) = 0$ and then finds $g(0) = 5$, which gives back the original input.

   a. What is wrong with Natalie's reasoning?

   A single point does not verify that the function is an inverse function. In order to be an inverse of the original function, we must have $f(g(x)) = g(f(x))$ for all x.

   b. Show that Natalie is incorrect by using other examples from the domain and range of f.

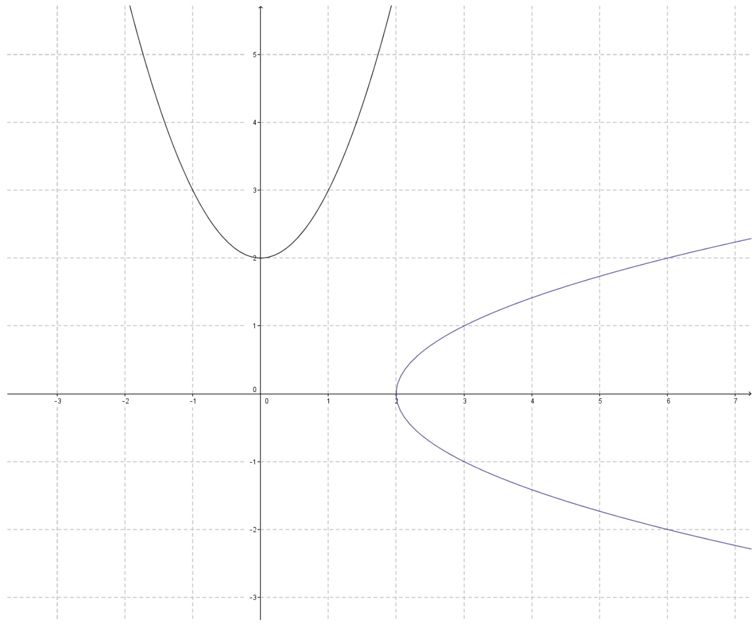
   Any other point will work; for instance, $f(0) = -5$ and $g(-5) = 10$.

   c. Find $f^{-1}(x)$. Where do f^{-1} and g intersect?

   $f^{-1}(x) = x + 5$
   The two functions intersect at the point $(0, 5)$, which explains why Natalie thought g was the inverse of f after she tried that point.

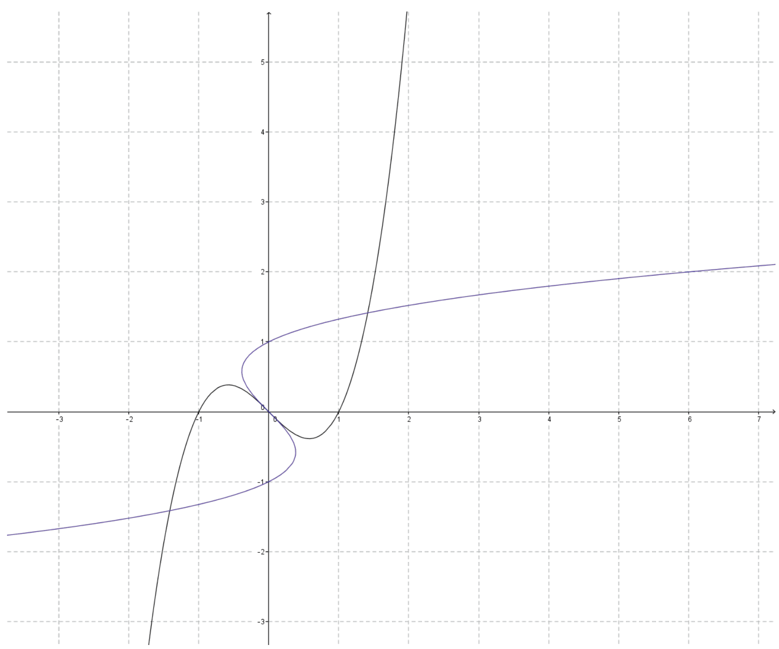
5. **Sketch a graph of the inverse of each function graphed below by reflecting the graph about the line $y = x$. State whether or not the inverse is a function.**

a.



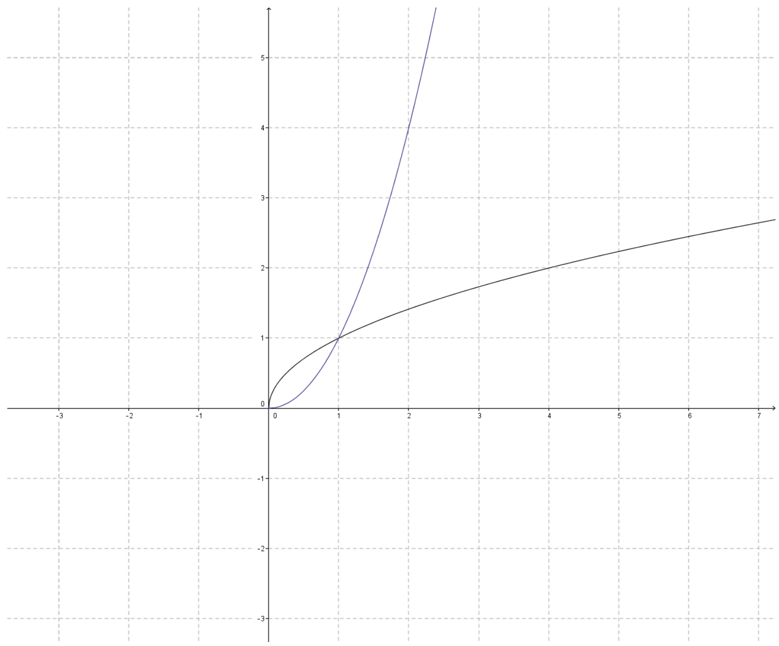
The reflected image is not a function.

b.



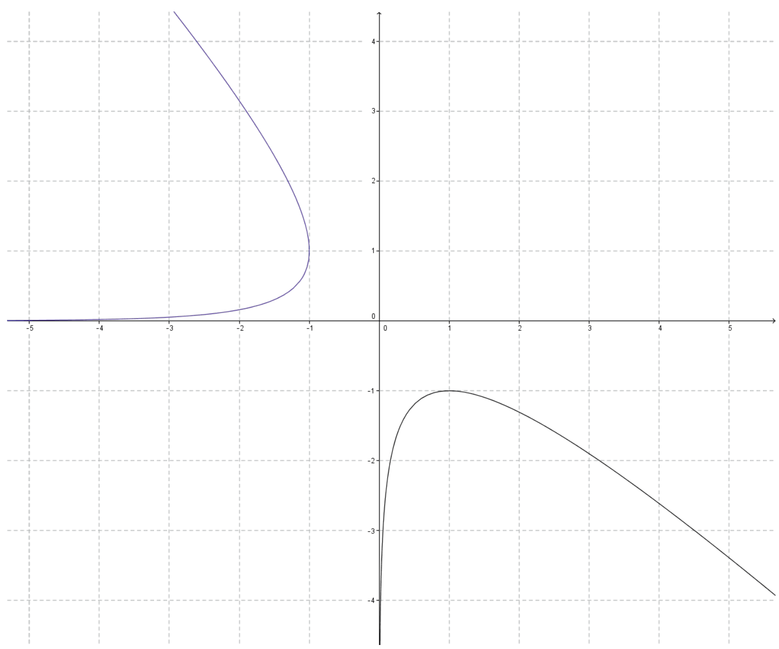
The reflected image is not a function.

c.



The reflected image is a function.

d.



The reflected image is not a function.

6. How can you tell before you reflect a graph over $y = x$ if its reflection will be a function or not?

If the function is not one-to-one, then its reflection will not be a function. Algebraically, if $f(x_1) = f(x_2)$ and $x_1 \neq x_2$, then the reflection will not be a function. Visually, this takes on the shape of a horizontal line intersecting the graph at two points. If any horizontal line can intersect the graph of a function at two points, then the reflection of the graph over the line $y = x$ will not be a function.

7. After finding several inverses, Callahan exclaims that every invertible linear function intersects its inverse at some point. What needs to be true about the linear functions that Callahan is working with for this to be true? What is true about linear functions that do not intersect their inverses?

The linear functions Callahan is working with must have slopes different from 1 in order to intersect with their inverses since this will guarantee that the functions will cross the $y = x$ line and that the inverses will intersect at $y = x$, if they intersect at all. If a linear function is parallel to $y = x$, then it will not intersect the graph of its inverse function.

8. If f is an invertible function such that $f(x) > x$ for all x, then what do we know about the inverse of f?

Since the function is always above the line $y = x$, we know that its inverse will always be below the line $y = x$. In other words, $f^{-1}(x) < x$. We can also see this by substituting $y = f(x)$, which gives us $y > x$. Switching x and y to find the inverse, we get $x > y$ or $x > f^{-1}(x)$.

9. Gavin purchases a new $\$2,995$ computer for his business, and when he does his taxes for the year, he is given the following information for deductions on his computer (this method is called MACRS—Modified Accelerated Cost Recovery System):

Period	Calculation for Deduction	Present Value
First Year	$D_1 = P_0 \,/\, 5 \times 200\% \times 50\%$	$P_0 - D_1 = P_1$
Second Year	$D_2 = P_1 \,/\, 5 \times 200\%$	$P_1 - D_2 = P_2$
Third Year	$D_3 = P_2 \,/\, 5 \times 200\%$	$P_2 - D_3 = P_3$

Where P_0 represents the value of the computer new.

a. Construct a table for the function D, giving the deduction Gavin can claim in year x for his computer, $x = \{1, 2, 3\}$.

x	1	2	3
$D(x)$	599	958.40	575.04

b. Find the inverse of D.

x	599	958.40	575.04
$D^{-1}(x)$	1	2	3

c. Construct a table for the function P, giving the present value of Gavin's computer in year x, $x = \{0, 1, 2, 3\}$.

x	0	1	2	3
$P(x)$	$2,995$	$2,396$	$1,437.60$	862.56

d. Find the inverse of P.

x	$2,995$	$2,396$	$1,437.60$	862.56
$P^{-1}(x)$	0	1	2	3

10. **Problem 9 used the MACRS method to determine the possible deductions Gavin could have for the computer he purchased. The straight-line method can be used also. Assume the computer has a salvage value of $\$500$ after 5 years of use; call this value S. Then, Gavin would be presented with this information when he does his taxes:**

Period	Calculation for Deduction	Present Value
First Year	$D_1 = (P_0 - S)/\ 5 \times 50\%$	$P_0 - D_1 = P_1$
Second Year	$D_2 = (P_0 - S)/\ 5$	$P_1 - D_2 = P_2$
Third Year	$D_3 = (P_0 - S)/\ 5$	$P_2 - D_2 = P_3$
Fourth Year	$D_4 = (P_0 - S)/\ 5$	$P_3 - D_3 = P_4$
Fifth Year	$D_5 = (P_0 - S)/\ 5$	S

a. **Construct a table for the function D, giving the deduction Gavin can claim in year x for his computer in $x = \{1, 2, 3, 4, 5\}$.**

x	1	2	3	4	5
$D(x)$	249.50	499	499	499	499

b. **What do you notice about the function for deduction in this problem compared to the function in Problem 9?**

The deduction values are a lot lower, and after the first year they are constant.

c. **If you are given the deduction that Gavin claims in a particular year using the straight-line method, is it possible for you to know what year he claimed it in? Explain. What does this tell us about the inverse of D?**

Unless it is the first year, you cannot tell the year in which Gavin claimed a particular deduction just by knowing the deduction amount. Gavin should claim $\$499$ as a deduction every year except for the first year.

Extension:

11. **For each function in Problem 1, verify that the functions are inverses by composing the function with the inverse you found (in each case, after applying both functions, you should end up with the original input).**

a.

$$f^{-1}(f(1)) = f^{-1}(3) = 1$$
$$f^{-1}(f(2)) = f^{-1}(15) = 2$$
$$f^{-1}(f(3)) = f^{-1}(8) = 3$$
$$f^{-1}(f(4)) = f^{-1}(-2) = 4$$
$$f^{-1}(f(5)) = f^{-1}(0) = 5$$

b.

$$\quad f \quad f^{-1}$$
$$0 \to 5 \to 0$$
$$2 \to 10 \to 2$$
$$4 \to 15 \to 4$$
$$6 \to 20 \to 6$$

c.

$$h(h^{-1}(5)) = h(1) = 5$$
$$h(h^{-1}(25)) = h(2) = 25$$
$$h(h^{-1}(125)) = h(3) = 125$$
$$h(h^{-1}(625)) = h(4) = 625$$

d.
$$\begin{array}{c} f^{-1} \quad f \\ 3 \to 1 \to 3 \\ 12 \to 2 \to 12 \\ 27 \to 3 \to 27 \\ 48 \to 4 \to 48 \end{array}$$

e.

x	3	6	12	24
$g^{-1}(x)$	-1	0	1	2
$g(g^{-1}(x))$	3	6	12	24

f.

x	1	10	100	$1,000$
$h(x)$	0	1	2	3
$h^{-1}(h(x))$	1	10	100	$1,000$

g. *Let $y = f(x)$.*
$$\begin{aligned} f^{-1}(f(x)) &= \frac{1}{2}(2x) \\ &= x \end{aligned}$$

h. *Let $y = f(x)$.*
$$\begin{aligned} f^{-1}(f(x)) &= 3\left(\frac{1}{3}x\right) \\ &= x \end{aligned}$$

i. *Let $y = f(x)$.*
$$\begin{aligned} f^{-1}(f(x)) &= (x-3)+3 \\ &= x \end{aligned}$$

j. *Let $y = f(x)$.*
$$\begin{aligned} f^{-1}(f(x)) &= -\frac{3}{2}\left(-\frac{2}{3}x+5\right)+\frac{15}{2} \\ &= x-\frac{15}{2}+\frac{15}{2} \\ &= x \end{aligned}$$

k. *Let $y = f(x)$; then $f(x) = \frac{2}{5}x - \frac{1}{5}$, $f^{-1}(x) = \frac{5}{2}x + \frac{1}{2}$.*
$$\begin{aligned} f^{-1}(f(x)) &= \frac{5}{2}\left(\frac{2}{5}x-\frac{1}{5}\right)+\frac{1}{2} \\ &= x-\frac{1}{2}+\frac{1}{2} \\ &= x \end{aligned}$$

l. *Let $y = f(x)$; then $f(x) = \frac{3}{7}x + 2$ and $f^{-1}(x) = \frac{7}{3}x - \frac{14}{3}$.*

$$\begin{aligned} f^{-1}(f(x)) &= \frac{7}{3}\left(\frac{3}{7}x + 2\right) - \frac{14}{3} \\ &= x + \frac{14}{3} - \frac{14}{3} \\ &= x \end{aligned}$$

m. *Let $y = f(x)$.*

$$\begin{aligned} f^{-1}(f(x)) &= \sqrt[3]{3(\frac{1}{3}(x-9)^3} + 9 \\ &= \sqrt[3]{(1)(x-9)^3} + 9 \\ &= x - 9 + 9 \\ &= x \end{aligned}$$

n. *Let $y = f(x)$.*

$$\begin{aligned} f^{-1}(f(x)) &= \frac{5}{3\left(\frac{5}{3x-4}\right)} + \frac{4}{3} \\ &= \frac{5}{3} \cdot \frac{3x-4}{5} + \frac{4}{3} \\ &= \frac{3x-4}{3} + \frac{4}{3} \\ &= \frac{3x-4+4}{3} \\ &= \frac{3x}{3} \\ &= x \end{aligned}$$

o. *Let $y = f(x)$.*

$$\begin{aligned} f^{-1}(f(x)) &= \sqrt[7]{\frac{1}{2}(2x^7+1) - \frac{1}{2}} \\ &= \sqrt[7]{x^7 + \frac{1}{2} - \frac{1}{2}} \\ &= \sqrt[7]{x^7} \\ &= x \end{aligned}$$

p. *Let $y = f(x)$.*

$$\begin{aligned} f^{-1}(f(x)) &= \left(\sqrt[5]{x}\right)^5 \\ &= x \end{aligned}$$

q. *Let $y = f(x)$.*

$$\begin{aligned}
f^{-1}\big(f(x)\big) &= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} \\
&= \frac{\frac{x+1}{x-1}+\frac{x-1}{x-1}}{\frac{x+1}{x-1}-\frac{x-1}{x-1}} \\
&= \frac{x+1+x-1}{x-1} \div \frac{x+1-(x-1)}{x-1} \\
&= \frac{2x}{x-1}\cdot\frac{x-1}{2} \\
&= x
\end{aligned}$$

Lesson 19: Restricting the Domain

Student Outcomes

- Students continue to read the inverse values of a function from a table and graph. They create the inverse of a function by solving an equation of the form $f(x) = y$.
- Students verify that two functions are inverses by composing them.
- Students choose a suitable domain to create an invertible function.

Lesson Notes

In this lesson, students work with functions and their inverses represented numerically, graphically, and algebraically. Lesson 18 helped students recall the meaning of the inverse of a function and the properties of inverse functions. This lesson builds on student understanding by providing several exercises that require students to create the inverse of a function by reading values from a table or a graph (**F-BF.B.4c**). Students consider that not every function has an inverse that is also a function. They consider how to restrict the domain of a function to produce an invertible function (**F-BF.B.4d**), setting the stage for the definition of the inverse trigonometric functions in Module 4. The lesson defines the adjective *invertible* as it applies to functions and provides practice for students to verify by composition that two functions are inverses (**F-BF.B.4b**).

Students should have access to graphing calculators or other graphing utilities during this lesson to aid their understanding.

Classwork

Opening Exercise (5 minutes)

Students should work this exercise independently and then compare answers with a partner. While circulating about the classroom, notice whether any concepts or vocabulary relating to functions and their inverses need to be retaught.

Students may need to be reminded that a function can be a simple mapping that assigns each element in the domain to a corresponding element in the range. The function f shown below pairs each element in the domain set with one element in the range.

Scaffolding:

- Create an anchor chart to post on the wall that includes the key information from the Lesson Summary in the previous lesson.
- Lead a short discussion of inverses that students have studied in the past. For example, adding 3 and subtracting 3 undo each other. Transformations also undo each other, and students studied inverse matrices.

Opening Exercise

The function f with domain $\{1, 2, 3, 4, 5\}$ is shown in the table below.

x	$f(x)$
1	7
2	3
3	1
4	9
5	5

a. What is $f(1)$? Explain how you know.

$f(1) = 7$ The table shows how the domain values and range values correspond for this function.

b. What is $f^{-1}(1)$? Explain how you know.

$f^{-1}(1) = 3$ Since this is the inverse function, the range values of f are the domain values of f^{-1}.

c. What is the domain of f^{-1}? Explain how you know.

The domain of f^{-1} is the range of f, so the domain of f^{-1} is the set of numbers $\{1, 3, 5, 7, 9\}$.

d. Construct a table for the function f^{-1}, the inverse of f.

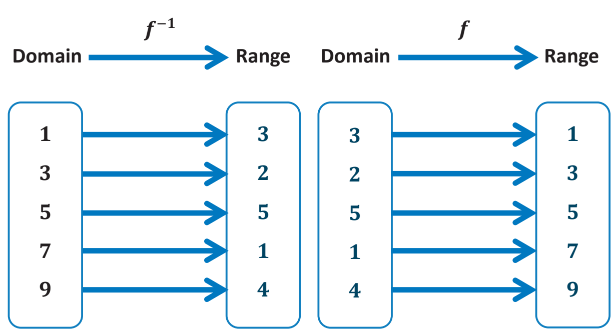
x	$f^{-1}(x)$
1	3
3	2
5	5
7	1
9	4

Exercises 1–5 (10 minutes)

After debriefing the Opening Exercise, have students continue to work in small groups on the next set of problems. If students are struggling to complete the mapping diagrams, consider modeling how to complete the first row as a whole class. Students may need a reminder about the operation of function composition. The function machine analogy is helpful for problems like these. Notice that the composite machine for a function and its inverse takes students back to the original domain values. Later in the lesson, students algebraically verify that one function is the inverse of another using function composition.

Exercises

1. **Complete the mapping diagram to show that $f\left(f^{-1}(x)\right) = x$.**

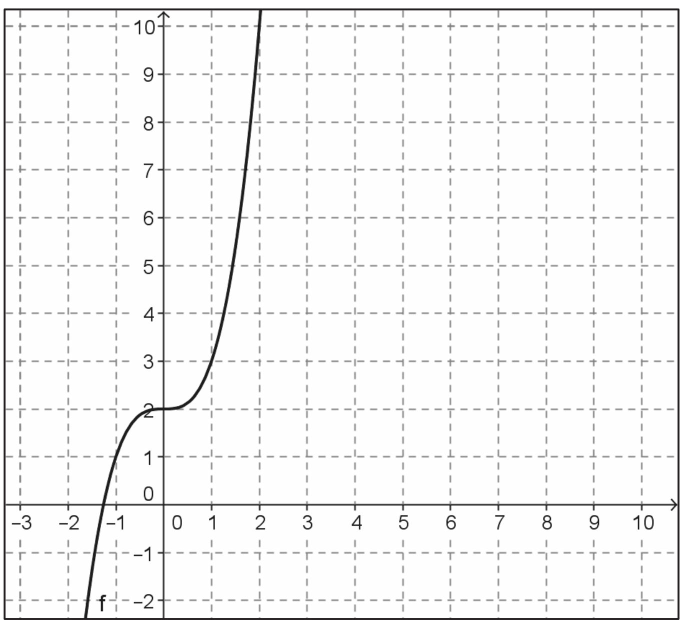


2. Complete the mapping diagram to show that $f^{-1}(f(x)) = x$.

Domain $\xrightarrow{f}$	Range	Domain $\xrightarrow{f^{-1}}$	Range
1	7	7	1
2	3	3	2
3	1	1	3
4	9	9	4
5	5	5	5

These next three exercises give students an opportunity to read values of an inverse function from a graph. Students also understand that the domain and range matter when creating the inverse of a function. Students may choose to graph the inverse by simply exchanging the x- and y-values of coordinates of points on the graph. They should also consider reflecting points across the line $y = x$. Be sure to discuss both approaches when debriefing these exercises.

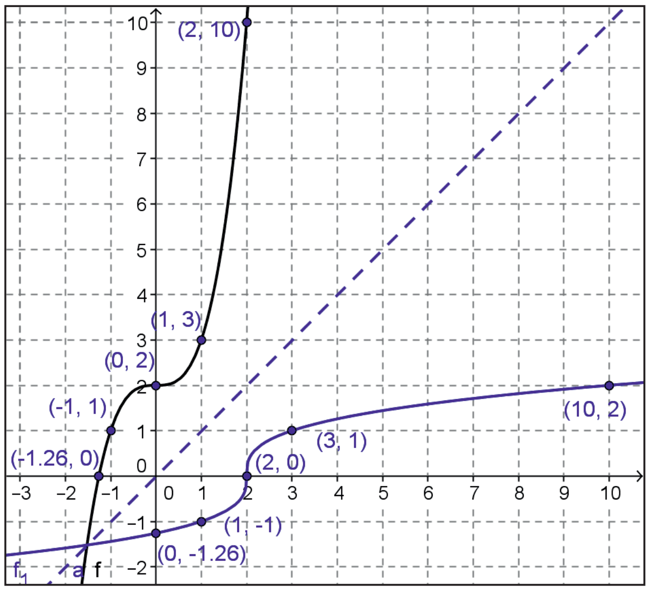
3. The graph of f is shown below.



a. Select several ordered pairs on the graph of f, and use those to construct a graph of f^{-1} in part b.

$(-1, 1)$, $(0, 2)$, $(1, 3)$, $(2, 10)$

b. Draw the line $y = x$, and use it to construct the graph of f^{-1} below.

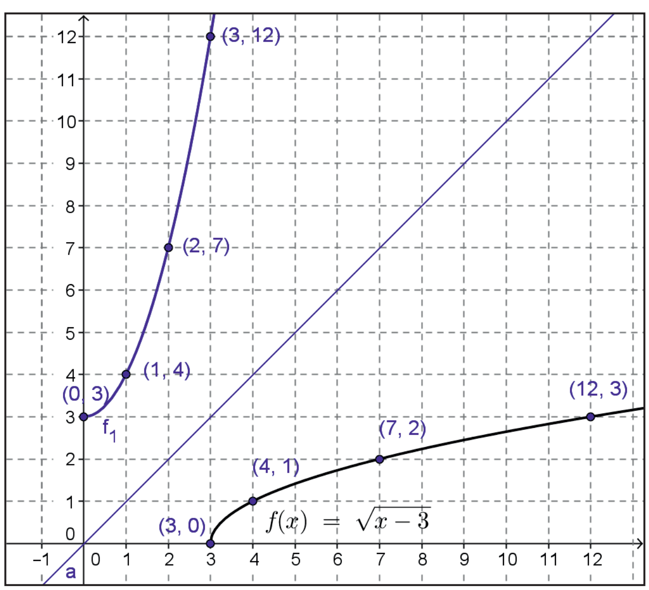


MP.3

c. The algebraic function for f is given by $f(x) = x^3 + 2$. Is the formula for $f^{-1}(x) = \sqrt[3]{x} - 2$? Explain why or why not.

No. The formula for f^{-1} is not correct because you don't get 0 when you substitute 2 into the formula for f^{-1}. The correct formula would be $f^{-1}(x) = \sqrt[3]{x-2}$.

4. The graph of $f(x) = \sqrt{x-3}$ is shown below. Construct the graph of f^{-1}.



5. **Morgan used the procedures learned in Lesson 18 to define $f^{-1}(x) = x^2 + 3$. How does the graph of this function compare to the one you made in Exercise 5?**

The graph of Morgan's function, $f^{-1}(x) = x^2 + 3$, is the inverse graph that we drew in Exercise 5. It has a domain that is assumed to be all real numbers. However, the inverse of f must have a domain equal to the range of f, which is $f(x) \geq 0$. Thus, the inverse's domain is not all real numbers but is instead restricted to $x \geq 0$. The graphs of these functions are identical for $x \geq 0$.

Discussion (5 minutes)

Lead a discussion as students share their thinking about Exercise 5. Many students do not consider that it is assumed the domain of a function is all real numbers unless specified otherwise. In order to define the function that is the inverse of f, the range of f must be used for the inverse's domain. Emphasize the importance of precise descriptions of the domain and range and why those are required by the definition of the inverse of a function given in Lesson 18.

MP.6 & MP.3

- When we create the inverse of a function, the domain and range are exchanged. What are the domain and range of f?
  - *The domain is $x \geq 3$, and the range is $f(x) \geq 0$.*
- What would be the domain and range of f^{-1}?
  - *The domain of f^{-1} would be the range of f, so the domain of f^{-1} is $x \geq 0$. The range of f^{-1} would be the domain of f, so the range of $f^{-1}$is $f(x) \geq 3$.*
- We cannot define the inverse of f algebraically without specifying its domain. The domain of f^{-1} is the range of f. Therefore, the domain of f^{-1} is $[0, \infty]$.
- How would you define f^{-1} for Exercise 5 algebraically?
  - $f^{-1}(x) = x^2 + 3, x \geq 0$
- Do you suppose that every function has an inverse that is a function? Explain your reasoning.
  - *If you reflected the graph of $f(x) = x^2 + 3$ across the line $y = x$ without restricting its domain, the points on the resulting graph would not meet the definition of a function.*
- In your own words, how do you define a function?
  - *Each element in the domain can be paired with only one element in the range.*
- Can you think of an example of a function that, when we exchange the domain and the range, would create a mathematical relationship that is not a function?
  - *If we started with $y = x^2$ whose graph corresponds to the graph of $f(x) = x^2$, then points on the graph of $x = y^2$ would NOT define a function. For example, $(1, 1)$ and $(1, -1)$ satisfy the equation $x = y^2$ but could not meet the definition of a function since the domain element 1 is paired with two range elements, 1 and -1.*

Exercises 6–8 (10 minutes)

The next exercises give students an opportunity to process the Discussion by practicing with functions that are NOT invertible. In Exercise 7, students begin to distinguish between invertible and non-invertible functions. In Lesson 10, students are introduced to the notion of a well-defined function as one that has a unique output for each input in the domain. Some books define these types of functions as one-to-one functions or strictly increasing or decreasing functions. These types of functions are invertible because there is never a domain value paired with more than one range value. Typical high school texts often refer to a vertical line test, which is used to determine whether a relation is a function, and a horizontal line test, which is used to determine whether a graph has an inverse that is also a function. This curriculum does not define either line test because the standards do not define relations formally. It may be tempting to help students by providing these tips or memory devices, but often they end up confusing students more than helping them. If these line tests are used, be sure to have students explain why the vertical or horizontal lines helped them to determine whether a graph represented a function or whether the inverse of a function represented graphically would also be a function.

6. **Construct the inverse of the function f given by the table below. Is the inverse a function? Explain your reasoning.**

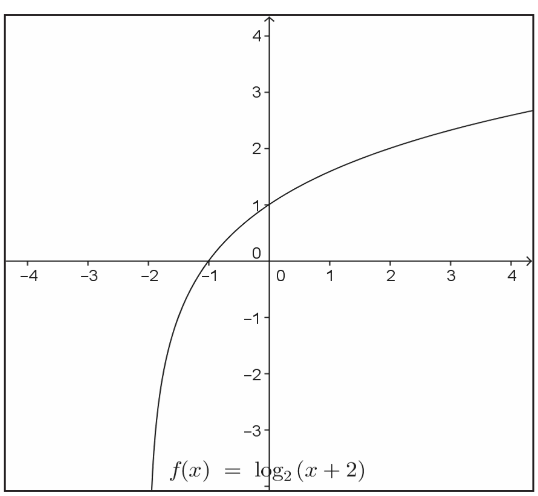
x	-3	-2	-1	0	1	2	3
$f(x)$	4	-1	-4	-5	-4	-1	4

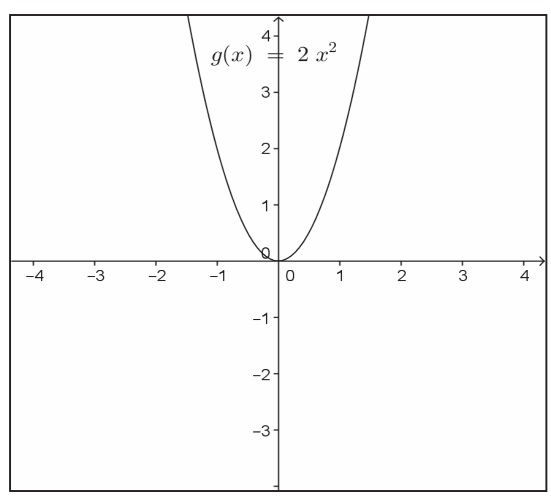
x	4	-1	-4	-5	-4	-1	4
Inverse of f	-3	-2	-1	0	1	2	3

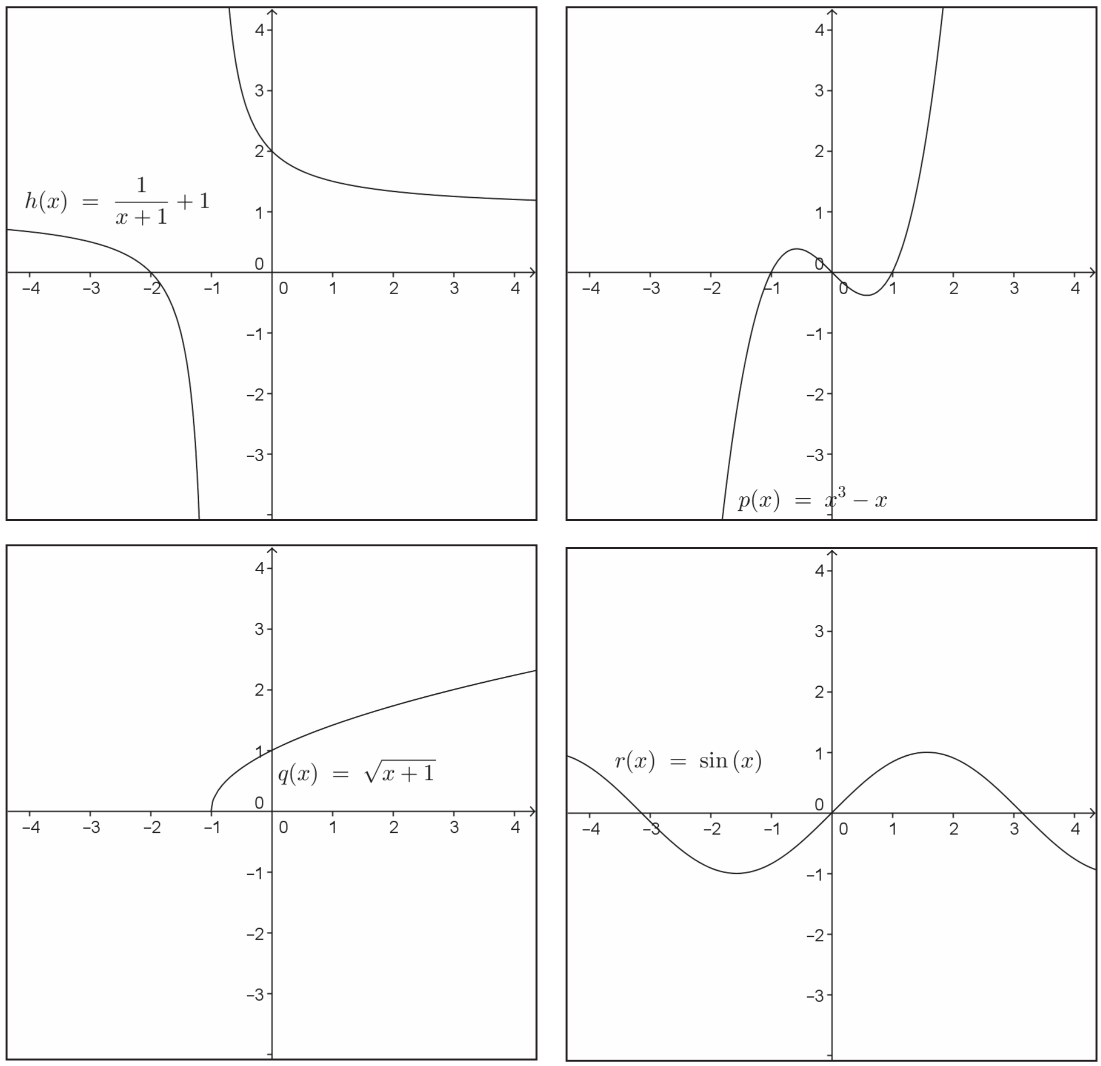
No, the inverse is not a function. Each element in the domain must have only one element in the range assigned to it. You can see that the number 4 is paired with -3 and 3.

- A function is said to be invertible if its inverse is also a function. Can you think of a quick way to tell whether a function is invertible?
  - *If an input value has two or more output values associated with it, then its inverse is not a function. This is easy to see when the function is graphed because you can look across the graph and see if two or more inputs give you the same output.*

7. **The graphs of several functions are shown below. Which ones are invertible? Explain your reasoning.**







The functions g, p, and r are not invertible. You can see that multiple elements of the domain are paired with a single range element. When the domain and range are exchanged to form the inverse, the result does not satisfy the definition of a function.

Exercise 8 may provide a challenge for some students. This exercise requires students to select a suitable domain for a function that makes it an invertible function and to create the inverse of the function. Let students wrestle with this problem, and encourage them to sketch the graph by hand or to use appropriate tools to help them such as a graphing calculator. Consider suggesting that students rewrite the expression in a different form that might make it easier to solve for x when y and x are exchanged. Prompt them to use the vertex form of a quadratic.

8. Given the function $f(x) = x^2 - 4$:

a. Select a suitable domain for f that makes it an invertible function. State the range of f.

A graph shows that the function f has a minimum point at $(0, -4)$. One possible domain would start with the x-value of the minimum, $x \geq 0$. The range of f given this domain would be $f(x) \geq -4$.

b. Write a formula for f^{-1}. State the domain and range of f^{-1}.

> *Scaffolding:*
> Have early finishers or students who wish an additional challenge repeat Exercise 8 with the function $f(x) = x^2 - 2x - 4$.

Since we know the vertex is $(0, -4)$, we can rewrite this function in vertex form and then create and solve an equation to find the inverse.

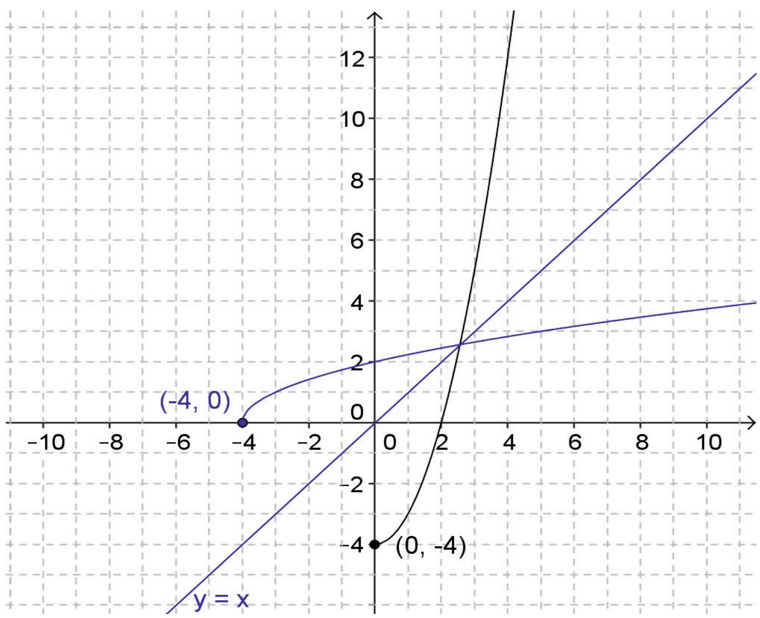
$$f(x) = x^2 - 4$$
$$y = x^2 - 4$$
$$x = y^2 - 4$$
$$x + 4 = y^2$$

$y = \sqrt{x+4}$ or $y = -\sqrt{x+4}$

However, select the positive branch because we want the range to be $y \geq 0$.

$y = \sqrt{x+4},\ x \geq -4$

c. Verify graphically that f, with the domain you selected, and f^{-1} are indeed inverses.



d. Verify that f and f^{-1} are indeed inverses by showing that $f(f^{-1}(x)) = x$ and $f^{-1}\left(f(x)\right) = x$.

$$\begin{aligned} f(f^{-1}(x)) &= \left(\sqrt{x+4}\right)^2 - 4 \\ &= x + 4 - 4 \\ &= x \end{aligned}$$

$$\begin{aligned} f^{-1}(f(x)) &= \sqrt{x^2 + 4 - 4} \\ &= \sqrt{x^2} \\ &= x,\ x \geq 0 \end{aligned}$$

Discuss the implications of part (d) above before moving on to the last exercises. In Exercise 1 and part (d) of Exercise 8, students verified by composition that one function was the inverse of another. Add this information to the anchor chart if there is one posted. Ask students to add this information to their notes as well.

COMPOSITION OF A FUNCTION AND ITS INVERSE: To verify that two functions are inverses, show that $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$.

Exercise 9 (8 minutes)

Students work with different types of functions to verify by composition that they are or are not inverses. Some students may approach the problem graphically, but when reviewing the solutions, be sure to explain that the algebraic approach is the preferred way to verify two functions are inverses unless proving by contradiction that they are not (e.g., by finding a point (a, b) on the graph of one function such that (b, a) is not a point on the graph of the alleged inverse).

9. **Three pairs of functions are given below. For which pairs are f and g inverses of each other? Show work to support your reasoning. If a domain is not specified, assume it is the set of real numbers.**

   a. $f(x) = \frac{x}{x+1}$, $x \neq -1$ **and** $g(x) = \frac{-x}{x-1}$, $x \neq 1$

$$\begin{aligned} f\big(g(x)\big) &= \frac{\frac{-x}{x-1}}{\frac{-x}{x-1}+1} \\ &= \frac{\frac{-x}{x-1}}{\frac{-x}{x-1}+\frac{x-1}{x-1}} \\ &= \frac{-x}{x-1}\cdot\frac{x-1}{-1} \\ &= x \end{aligned}$$

$$\begin{aligned} g\big(f(x)\big) &= \frac{-\frac{x}{x+1}}{\frac{x}{x+1}-1} \\ &= \frac{-\frac{x}{x+1}}{\frac{x}{x+1}-\frac{x+1}{x+1}} \\ &= -\frac{x}{x+1}\cdot\frac{x+1}{-1} \\ &= x \end{aligned}$$

   The two functions are inverses.

   b. $f(x) = \sqrt{x} - 1$, $x \geq 0$ **and** $g(x) = (x+1)^2$

$$f\big(g(x)\big) = \sqrt{(x+1)^2} - 1 = x + 1 - 1 = x$$
$$g\big(f(x)\big) = \left(\sqrt{x} - 1 + 1\right)^2 = \sqrt{x}^2 = x$$

   These two functions are inverses as long as we restrict the domain of g to be $x \geq -1$ since the range of f is $f(x) \geq -1$.

c. $f(x) = -0.75x + 1$ and $g(x) = -\frac{4}{3}x - \frac{4}{3}$

$$f(g(x)) = -\frac{3}{4}\left(-\frac{4}{3}x - \frac{4}{3}\right) + 1 = x + 1 + 1 = x + 2$$

These two functions are not inverses.

Closing (3 minutes)

Have students respond to the following questions individually, in writing, or with a partner to close this lesson. The question answers are sample student responses.

- What are two things you learned about a function and its inverse function?
  - *The two functions undo each other when they are composed. You may need to restrict the domain of a function to make its inverse a function.*
- What is a question you still have about a function and its inverse function?
  - *Do all functions have inverses? How do you choose a suitable domain for a trigonometric function?*

The Lesson Summary can be used to clarify any questions students may still have.

Lesson Summary

COMPOSITION OF A FUNCTION AND ITS INVERSE: To verify that two functions are inverses, show that $f(g(x)) = x$ and $g(f(x)) = x$.

INVERTIBLE FUNCTION: The domain of a function f can be restricted to make it invertible. A function is said to be invertible if its inverse is also a function.

Exit Ticket (4 minutes)

Name ______________________________ Date ______________

Lesson 19: Restricting the Domain

Exit Ticket

Let $f(x) = x^2 - 3x + 2$.

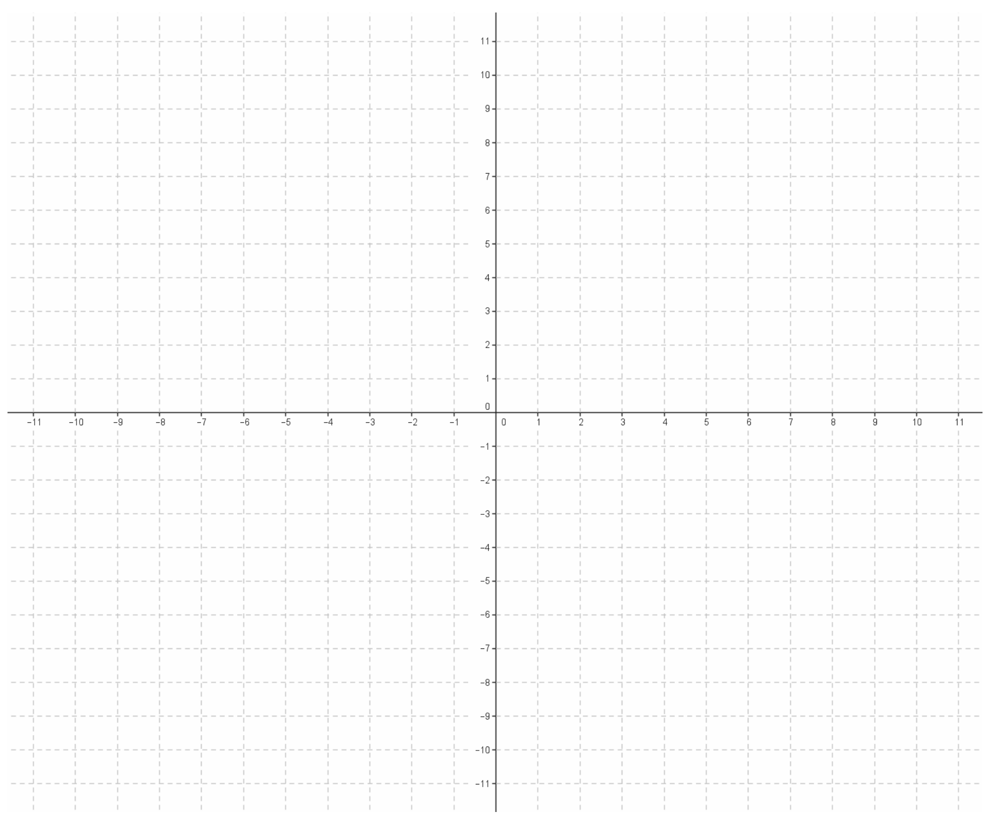
a. Give a restricted domain for f where it is invertible.

b. Find the inverse of f for the domain you gave in part (a).

c. State the domain and range of the function you found in part (b).

d. Verify through function composition that the function you found in part (b) is the inverse of f.

e. Graph both functions on the domains specified.



Exit Ticket Sample Solutions

Let $f(x) = x^2 - 3x + 2$.

a. **Give a restricted domain for f where it is invertible.**

The vertex of this parabola is at $x = \frac{3}{2}$, so either $\left(-\infty, \frac{3}{2}\right]$ or $\left[\frac{3}{2}, \infty\right)$ would be acceptable. The remaining answers assume $\left[\frac{3}{2}, \infty\right)$ was chosen.

b. **Find the inverse of f for the domain you gave in part (a).**

$$\begin{aligned} f(x) &= (x - 1.5)^2 - 0.25 \\ x &= (y - 1.5)^2 - 0.25 \\ y &= \pm\sqrt{x + 0.25} + 1.5 \end{aligned}$$

For $f(x)$ on the domain $\left[\frac{3}{2}, \infty\right)$, $f^{-1}(x) = \sqrt{x + 0.25} + 1.5$.

c. **State the domain and range of the function you found in part (b).**

For f^{-1}, the domain is all real numbers greater than or equal to -0.25, and the range is all real numbers greater than or equal to 1.5.

d. **Verify through function composition that the function you found in part (b) is the inverse of f.**

For $x \geq -0.25$, we have:

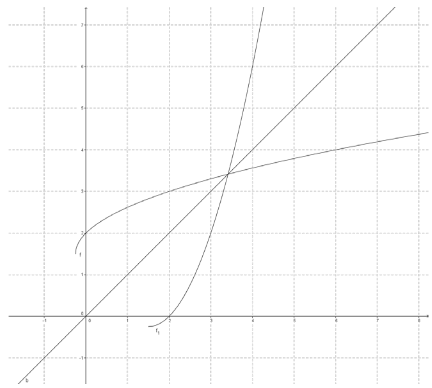
$$\begin{aligned} f\left(f^{-1}(x)\right) &= f\left(\sqrt{x + 0.25} + 1.5\right) \\ &= \left(\sqrt{x + 0.25} + 1.5\right)^2 - 3\left(\sqrt{x + 0.25} + 1.5\right) + 2 \\ &= x + 0.25 + 3\sqrt{x + 0.25} + 2.25 - 3\sqrt{x + 0.25} - 4.5 + 2 \\ &= x \end{aligned}$$

For $x \geq 1.5$, we have:

$$\begin{aligned} f^{-1}\left(f(x)\right) &= f^{-1}(x^2 - 3x + 2) \\ &= \sqrt{x^2 - 3x + 2 + 0.25} + 1.5 \\ &= \sqrt{x^2 - 3x + \frac{9}{4}} + \frac{3}{2} \\ &= \sqrt{\left(x - \frac{3}{2}\right)^2} + \frac{3}{2} \\ &= x - \frac{3}{2} + \frac{3}{2} \\ &= x \end{aligned}$$

Thus, $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$.

e. **Graph both functions on the domains specified.**



Problem Set Sample Solutions

1. **Let f be the function that assigns to each student in the class her biological mother.**
   a. **In order for f to have an inverse, what condition must be true about students in your class?**

      No students in the class can share the same biological mother.

   b. **If we enlarged the domain to include all students in the school, would this larger domain function have an inverse? Explain.**

      Probably not. Most schools contain siblings.

2. **Consider a linear function of the form $f(x) = mx + b$, where m and b are real numbers, and $m \neq 0$.**
   a. **Explain why linear functions of this form always have an inverse that is also a function.**

      The graphs of these functions are lines. When a line is reflected over the line $y = x$, the image is also a line and, therefore, can be represented as a linear function.

   b. **State the general form of a line that does not have an inverse.**

      $y = k$ for some real number k That is, only linear functions whose graphs are horizontal lines do not have inverse functions.

   c. **What kind of function is the inverse of an invertible linear function (e.g., linear, quadratic, exponential, logarithmic, rational)?**

      The inverse of an invertible linear function would also be a linear function.

   d. **Find the inverse of a linear function of the form $f(x) = mx + b$, where m and b are real numbers, and $m \neq 0$.**

$$x = my + b$$
$$y = \frac{x - b}{m}$$

So, $f^{-1}(x) = \frac{x-b}{m}$ for any linear function $f(x) = mx + b$, with $m \neq 0$.

3. Consider a quadratic function of the form $f(x) = b\left(\frac{x-h}{a}\right)^2 + k$ for real numbers a, b, h, k, and $a, b \neq 0$.

   a. Explain why quadratic functions never have an inverse without restricting the domain.

   Every quadratic function is represented by the graph of a parabola, which always reflects over the $y = x$ line in such a way that one input maps to two outputs, violating the definition of a function. Thus, a quadratic function could only have an inverse if its domain is restricted.

   b. What are the coordinates of the vertex of the graph of f?

   (h, k)

   c. State the possible domains you can restrict f on so that it has an inverse.

   The function f on the domains $(-\infty, h]$ and $[h, \infty)$ is invertible.

   d. What kind of function is the inverse of a quadratic function on an appropriate domain?

   The inverse of a quadratic function on an appropriate domain is a square root function.

   e. Find f^{-1} for each of the domains you gave in part (c).

$$x = b\left(\frac{y-h}{a}\right)^2 + k$$
$$\frac{x-k}{b} = \left(\frac{y-h}{a}\right)^2$$
$$y = \pm a\sqrt{\frac{x-k}{b}} + h$$

   The inverse function is either described by $y = a\sqrt{\frac{x-k}{b}} + h$ or $y = -a\sqrt{\frac{x-k}{b}} + h$ depending on which domain is chosen for f.

4. Show that $f(x) = mx + b$ for real numbers m and b with $m \neq 0$ has an inverse that is also a function.

$$y = mx + b$$
$$x = my + b$$
$$x - b = my$$
$$\frac{1}{m}x - \frac{b}{m} = y$$

Thus, $f^{-1}(x) = \frac{1}{m}x - \frac{b}{m}$, which is a linear function. One can see from the graph of a line that each input in the domain is paired with one output.

5. **Explain why $f(x) = a(x-h)^2 + k$ for real numbers a, h, and k with $a \neq 0$ does not have an inverse that is a function. Support your answer in at least two different ways (numerically, algebraically, or graphically).**

$$y = a(x-h)^2 + k$$
$$x = a(y-h)^2 + k$$
$$\frac{x-k}{a} = (y-h)^2$$

This equation has two solutions when you take the square root:

$$y - h = \sqrt{\frac{x-k}{a}} \text{ or } y - h = -\sqrt{\frac{x-k}{a}}$$

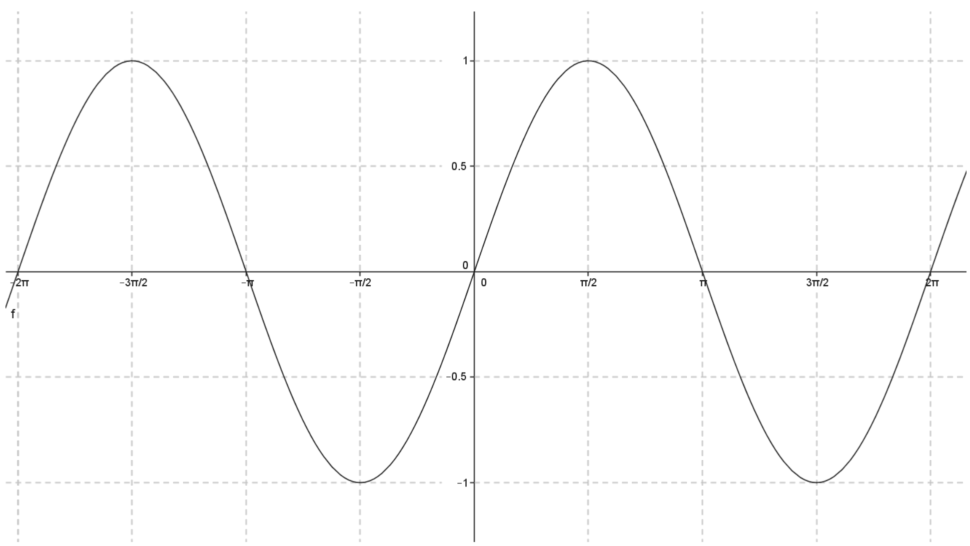
You can see that selecting a single value for x results in two corresponding values of y for all $x \neq k$. Thus, the inverse is not a function.

Graphically, the graph of f is a quadratic function with a vertex at (h, k). The symmetry of this graph means that there are two domain values with the same range value for all $x \neq h$. When the graph of this function is reflected over the line $y = x$, the resulting graph does not meet the definition of a function.

Extension:

6. **Consider the function $f(x) = \sin(x)$.**

   a. **Graph $y = f(x)$ on the domain $[-2\pi, 2\pi]$.**



   b. **If we require a restricted domain on f to be continuous and cover the entirety of the range of f, how many possible choices for a domain are there in your graph from part (a)? What are they?**

   There are three possible choices: $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

   c. **Make a decision on which restricted domain you listed in part (b) makes the most sense to choose. Explain your decision.**

   Answers may vary, but it is expected most students choose $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ since it includes the origin.

d. **Use a calculator to evaluate $\sin^{-1}(0.75)$ to three decimal places. How can you use your answer to find other values ψ such that $\sin(\psi) = 1$? Verify that your technique works by checking it against your graph in part (a).**

$$\sin^{-1}(0.75) \approx 0.848$$

On the unit circle, sine values are equal for supplementary angles of rotation, so $\pi - 0.848$ gives another approximate result. Infinitely many values can be found from these two values by adding integer multiples of 2π to either value.

Lesson 20: Inverses of Logarithmic and Exponential Functions

Student Outcomes

- In order to demonstrate understanding of the inverse relationship between exponents and logarithms, students construct the inverse of exponential and logarithmic functions from a table, a graph, or an algebraic representation.
- Students compose functions to verify that exponential functions and logarithmic functions are inverses.

Lesson Notes

This lesson focuses on logarithmic and exponential functions. Students have worked with exponential functions since Algebra I, and logarithmic functions were introduced and studied extensively in Algebra II. The inverse of an exponential function was first defined in Algebra II as students solved equations of the form $a \cdot b^{cx} = d$ and came to understand that the solution to this type of equation is a logarithm (**F-LE.B.5**). This lesson reviews what students learned in Algebra II about the inverse relationship between exponents and logarithms (**F-BF.B.5**) and uses composition to verify that a logarithmic function and an exponential function are inverses (**F-BF.B.4d**).

Depending on how much students recall from Algebra II, it may be necessary to provide some review and practice for working with exponential and logarithmic expressions and rewriting them using their definitions, identities, and properties. Pertinent vocabulary and definitions are included at the end of this lesson.

This lesson is greatly enhanced by the use of technology. Students should have access to graphing calculators or other graphing utilities. If access is limited, then teacher demonstrations can be utilized instead at those points where technology is infused in the lesson.

Classwork

Opening (2 minutes)

Explain briefly to students that they are going to continue working to understand the inverses of other functions that they have studied in the past.

- What are some of the different types of functions we have studied so far this year and in past years?
  - *We have studied polynomial, rational, exponential, logarithmic, and trigonometric functions to name a few.*

Tell them that the focus of this lesson is on exponential and logarithmic functions and that Module 4 considers the inverse functions of trigonometric functions as well.

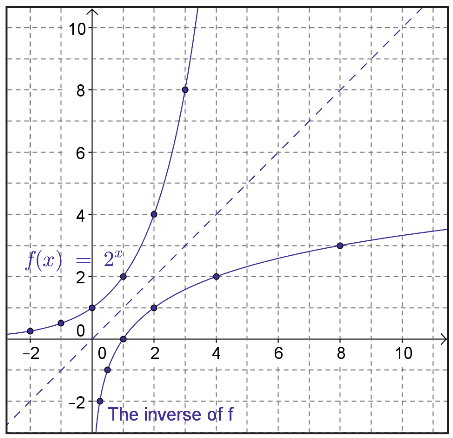
Opening Exercise (5 minutes)

Students should complete the Opening Exercise individually and then compare their answers with a partner. How students do on these problems gives the teacher insight into how much reteaching and reviewing may be necessary during this lesson. For example, if a majority of students do not recognize quickly that the inverse of $f(x) = 2^x$ is a logarithmic function, it is necessary to provide additional support while moving through this lesson. Be careful not to jump in too quickly though. Give students time to think through and recall what they have already learned.

Opening Exercise

Let $f(x) = 2^x$.

a. **Complete the table, and use the points $(x, f(x))$ to create a sketch of the graph of $y = f(x)$.**



x	$f(x)$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

b. **Create a table of values for the function f^{-1}, and sketch the graph of $y = f^{-1}(x)$ on the grid above.**

The inverse is sketched above.

x	$f^{-1}(x)$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

MP.2 & MP.7

c. **What type of function is f^{-1}? Explain how you know.**

It appears to be a logarithmic function. We are plotting powers of 2 on the horizontal axis and the corresponding exponents on the vertical axis. This is how we define a logarithmic function.

Example 1 (5 minutes)

This example is intended to show students that the inverse of $f(x) = 2^x$ really is a logarithm. If students readily recalled that the inverse of the function in the Opening Exercise should be the graph of a logarithmic function, then students could work Example 1 in small groups. Use the discussion as needed to help students recall the definition of a logarithm from Algebra II. The definition of a logarithm is stated below:

> **LOGARITHM**: If three numbers L, b, and x are related by $x = b^L$, then L is the logarithm base b of x, and we write $\log_b(x)$. That is, the value of the expression $L = \log_b(x)$ is the power of b needed to obtain x. Further, b must be a real number such that $0 < b < 1$ or $b > 1$.

Use these questions to guide students on this Example.

- How do we algebraically find the inverse of a function?
  - *Replace $f(x)$ with y, exchange the x and y symbols, and solve for y.*
- How do we undo an exponentiation?
  - *You need to write the equation in logarithm form.*
- What is the definition of a logarithm?
  - *For three numbers L, b, and x, $x = b^L$ if and only if $L = \log_b(x)$.*

Example

Given $f(x) = 2^x$, use the definition of the inverse of a function and the definition of a logarithm to write a formula for $f^{-1}(x)$.

$$y = 2^x$$

If g is the inverse of f, then $y = f(x)$ implies that $g(y) = x$, so you exchange the x and y variables.

$$x = 2^y$$

We use this definition of logarithm to rewrite $x = 2^y$ to be

$$\log_2(x) = y$$
$$y = \log_2(x)$$
$$f^{-1}(x) = \log_2(x)$$

Consider having students use a graphing calculator or other graphing utility to verify that the values in the table in the Opening Exercise part (b) correspond to points on the graph of $y = \log_2(x)$.

Scaffolding:

To provide a more concrete approach, give students specific numeric examples.

- Show by composition that the following pairs of functions are inverses:
  - $f(x) = 2^x$ and $g(x) = \log_2(x)$
  - $f(x) = 3^x$ and $g(x) = \log_3(x)$
  - $f(x) = 2x + 1$ and $g(x) = \frac{1}{2}(x - 1)$.
- Then, have them show that the general form of a linear function $f(x) = mx + b$ and $g(x) = \frac{1}{m}(x - b)$ are inverses.

Exercises 1–10 (15 minutes)

In Algebra II, students did not know about function composition. Here they use the definition of logarithm to prove the inverse relationship between logarithms and exponents. Begin with numeric exercises.

Exercises

1. **Find the value of y in each equation. Explain how you determined the value of y.**

   a. $y = \log_2(2^2)$

   $y = 2$ because the logarithm of 2^2 is the exponent to which you would raise 2 to get 2^2.

   b. $y = \log_2(2^5)$

   $y = 5$ because the logarithm of 2^5 is the exponent to which you would raise 2 to get 2^5.

   c. $y = \log_2(2^{-1})$

   $y = -1$ because the logarithm of 2^{-1} is the exponent to which you would raise 2 to get 2^{-1}.

   d. $y = \log_2(2^x)$

   $y = x$ because the logarithm of 2^x is the exponent to which you would raise 2 to get 2^x.

2. **Let $f(x) = \log_2(x)$ and $g(x) = 2^x$.**

   a. **What is $f\big(g(x)\big)$?**

   $f\big(g(x)\big) = \log_2(2^x)$

   Thus, $f\big(g(x)\big) = x$.

   b. **Based on the results of part (a), what can you conclude about the functions f and g?**

   The two functions would be inverses.

3. **Find the value of y in each equation. Explain how you determined the value of y.**

   a. $y = 3^{\log_3(3)}$

   $y = 3$ because $\log_3(3) = 1$, and by substituting, we get $y = 3^1 = 3$.

   b. $y = 3^{\log_3(9)}$

   $y = 9$ because $\log_3(9) = 2$, and by substituting, we get $y = 3^2 = 9$.

   c. $y = 3^{\log_3(81)}$

   $y = 81$ because $\log_3(81) = 4$, and by substituting, we get $y = 3^4 = 81$.

   d. $y = 3^{\log_3(x)}$

   $y = x$ because if we rewrite the equation in logarithm form, we get $\log_3(y) = \log_3(x)$, which shows that $y = x$.

4. Let $f(x) = \log_3(x)$ and $g(x) = 3^x$.

   a. What is $g(f(x))$?

   $g(f(x)) = 3^{\log_3(x)}$

   Thus, $g(f(x)) = x$.

   b. Based on the results in part (a), what can you conclude about the functions f and g?

   The functions f and g are inverses.

MP.3

5. Verify by composition that the functions $f(x) = b^x$ and $g(x) = \log_b(x)$ for $b > 0$ are inverses of one another.

   We need to show that $f(g(x)) = b^{\log_b(x)} = x$.

   Let $y = b^{\log_b(x)}$.

   Then, using the definition of logarithm,

   $\log_b(y) = \log_b(x)$,

   which means that $y = x$. Substituting into the equation above,

   $x = b^{\log_b(x)}$.

   We also need to show that $g(f(x)) = \log_b(b^x) = x$.

   Let $y = \log_b(b^x)$.

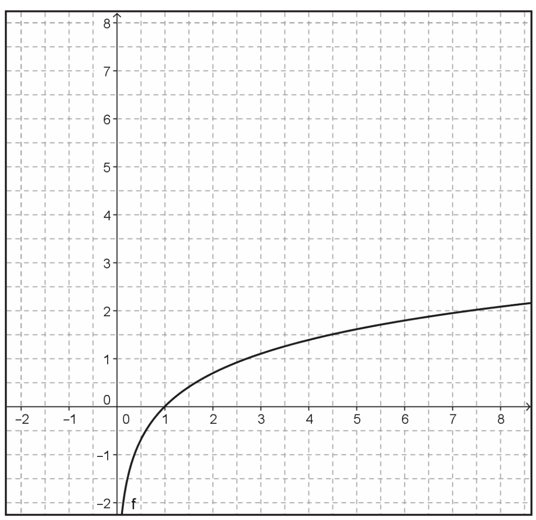
   Then, using the definition of logarithm,

   $b^y = b^x$,

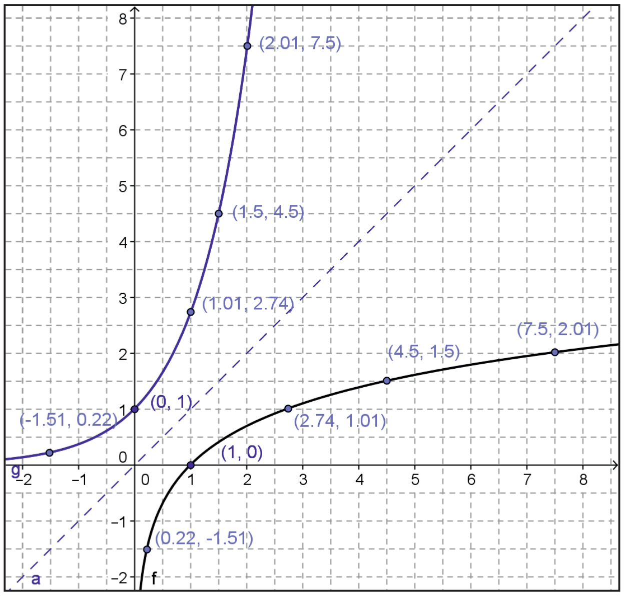
   which means that $y = x$. Substituting into the equation above,

   $x = \log_b(b^x)$.

6. The graph of $y = f(x)$, a logarithmic function, is shown below.



a. **Construct the graph of $y = f^{-1}(x)$.**



b. **Estimate the base b of these functions. Explain how you got your answer.**

$f(b) = 1$ gives a value for $b \approx 2.75$. Thus, the base appears to be about 2.75 because $\log_b b = 1$ for any base b. You can see on the exponential graph that x is 1, and the corresponding y-value is 2.75.

The graph of the logarithmic function provided in the student materials is the graph of $y = \ln(x)$, and the graph of the solution shown above is the graph of $y = e^x$. Do not share this information with students yet; they investigate the base of these two functions in the next few exercises by comparing the value of e from their calculators with the base they estimated in Exercise 6 part (b). Then, they graph the functions $y = e^x$ and $y = \ln(x)$ on their calculators and graph the composition of these two functions, which is $y = x$.

If access to technology is limited in the classroom, the next four exercises can be done as a demonstration. After each exercise, allow time for students to respond to and process what they are seeing as the teacher demonstrates the problems on a graphing calculator or using other graphing technology. Have them share their responses with a partner or small group before discussing the answers with the whole class.

The information on the following page is provided for additional teacher background information or as an extension to this lesson.

The value of e can also be estimated using a series. The number e is equivalent to the infinite series shown below. This formula comes from the Maclaurin Series for e^x that students see if they continue on to study calculus. If students are unfamiliar with the factorial notation, briefly introduce it. Summing the first several terms of this series gives an estimate for the value of e.

An infinite series can be used to define the irrational number,

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!} + \cdots,$$

where n is a whole number.

Note: For positive integers n, the value of <u>n factorial</u> denoted $n!$ is $n(n-1)(n-2)(n-3)\ldots(1)$. Thus, $3! = 3 \cdot 2 \cdot 1 = 6$, and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

In the series above, since n is a whole number, the first term would be $\frac{1}{0!}$, which is equal to 1 because $0!$ is defined to be 1. Interestingly, $1!$ is also equal to 1.

To approximate e using this series on a graphing calculator, students can quickly generate a table of values for the first n terms of this series equal to e by typing $Y1 = \text{sum}(\text{seq}(1/X!, X, 0, X, 1))$ into a graphing calculator and viewing the values of $Y1$ in the table feature of the calculator.

7. **Use a calculator to get a very accurate estimate of the irrational number e.**

   On the calculator, $e \approx 2.718\,281\,828\,46$.

8. **Is the graph of $y = f^{-1}(x)$ in Exercise 6 a good approximation of the function $g(x) = e^x$? Explain your reasoning.**

   We estimated that the base was 2.75, so it is close to the value of e.

9. **Show that $f(x) = \ln(x)$ and $g(x) = e^x$ are inverse functions by graphing $y = f(g(x))$ and $y = g(f(x))$ on a graphing calculator. Explain how your graphs support the fact that these two functions are indeed inverses of one another.**

   When graphed, the graphs of $y = f(g(x))$ and $y = g(f(x))$ are the same as the graph of $y = x$ for all values of x.

Scaffolding:

For struggling learners, create a summary chart of the properties of logarithms and exponents that can be found in Algebra II Module 3 Lesson 4 and Lesson 12.

Provide fluency practice with rewriting expressions using the properties. A rapid white board technique may be used to do this. Work on expanding expressions (e.g., $\log\left(\frac{2x}{x-1}\right) = \log(2) + \log(x) - \log(x-1)$) as well as condensing expressions (e.g., $(2^3 \cdot 2^x)^2 = 2^{6+2x}$).

On a graphing calculator, if students enter the following into $\boxed{Y =}$

$$Y1 = \ln(x)$$
$$Y2 = e^\wedge(x)$$
$$Y3 = Y1(Y2(x))$$
$$Y4 = Y2(Y1(x))$$

and then graph these equations, they see that the graphs of $Y3$ and $Y4$ are the graph of the equation $y = x$, which verifies by composition that these two functions are inverses of one another.

10. What is the base of the natural logarithm function $f(x) = \ln(x)$? Explain how you know.

Based on the results of Exercise 9, we can conclude that the base of the natural logarithm function is the irrational number e.

Exercises 11–12 (10 minutes)

In these problems, students work with exponential and logarithmic functions to algebraically find a formula for the inverse of each function. Exercise 12 asks them to verify by graphing that the functions are inverses. If access to technology is limited, take time to model different solutions as a whole class. Be sure to demonstrate at least one incorrect response so students can see that the reflection property does not hold.

The same process that was used to algebraically find the inverse of a function in Lessons 18 and 19 can be applied to find the inverses of exponential and logarithmic functions.

- How is this definition similar to the definition of inverse functions?
  - They both involve switching the x and the y in a way.

11. Find the inverse of each function.

a. $f(x) = 2^{x-3}$

$$\begin{aligned} y &= 2^{x-3} \\ x &= 2^{y-3} \\ \log_2(x) &= y - 3 \\ y &= 3 + \log_2(x) \\ f^{-1}(x) &= 3 + \log_2(x) \end{aligned}$$

b. $g(x) = 2\log(x-1)$

$$\begin{aligned} y &= 2\log(x-1) \\ x &= 2\log(y-1) \\ \frac{x}{2} &= \log(y-1) \\ 10^{\frac{x}{2}} &= y - 1 \\ y &= 10^{\frac{x}{2}} + 1 \\ g^{-1}(x) &= 10^{\frac{x}{2}} + 1 \end{aligned}$$

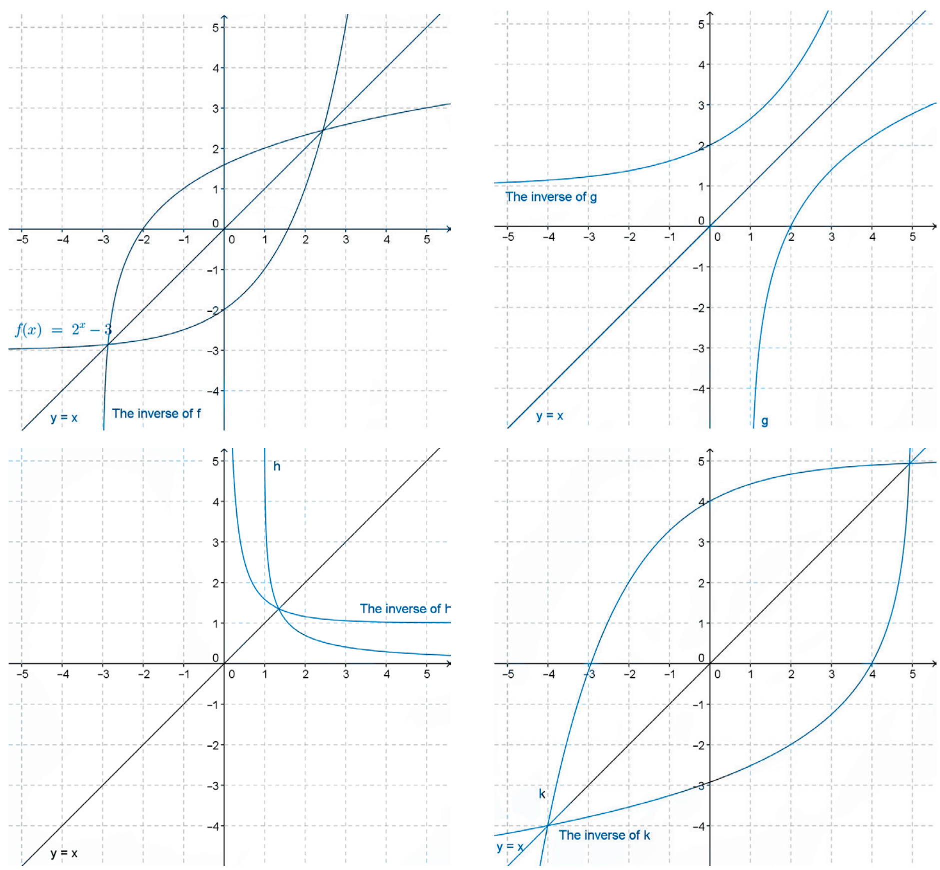
c. $h(x) = \ln(x) - \ln(x-1)$

$$\begin{aligned} y &= \ln(x) - \ln(x-1) \\ x &= \ln(y) - \ln(y-1) \\ x &= \ln\left(\frac{y}{y-1}\right) \\ e^x &= \frac{y}{y-1} \\ e^x(y-1) &= y \\ e^x y - y &= e^x \\ y(e^x - 1) &= e^x \\ y &= \frac{e^x}{e^x - 1} \\ h^{-1}(x) &= \frac{e^x}{e^x - 1} \end{aligned}$$

d. $k(x) = 5 - 3^{-\frac{x}{2}}$

$$y = 5 - 3^{-\frac{x}{2}}$$
$$x = 5 - 3^{-\frac{y}{2}}$$
$$5 - x = 3^{-\frac{y}{2}}$$
$$\log_3(5 - x) = -\frac{y}{2}$$
$$-2\log_3(5 - x) = y$$
$$k^{-1}(x) = -2\log_3(5 - x)$$

12. **Check your solutions to Exercise 11 by graphing the functions and the inverses that you found and verifying visually that the reflection property holds.**



Students should use technology to check the graphs in Exercise 12. If none is available, have them check their solutions by selecting a few values of x and finding the corresponding range element y and then showing that if (x, y) is on the graph of the function, then (y, x) would be on the graph of the inverse.

Closing (3 minutes)

To close this lesson, have students respond in writing or with a partner to the questions below.

- Explain using concepts of inverse functions why $b^{\log_b(x)} = x$ and $\log_b(b^x) = x$.
  - *Since the functions $f(x) = b^x$ and $g(x) = \log_b(x)$ are inverses of one another, when you compose them you simply get x. The identities above represent the composition of these functions, and due to the inverse nature of logarithms and exponents, the resulting composition of the two functions is equal to x.*
- Explain why all exponential functions of the form $f(x) = b^x$ are invertible.
  - *These functions have exactly one y-value for each x-value, so when the domain and range sets are exchanged, the inverses also have exactly one x-value in the domain for each y-value in the range and, therefore, are a function without having to restrict the domain and range of the original function.*
- Explain why all logarithmic functions of the form $f(x) = \log_b(x)$ are invertible.
  - *These functions have exactly one y-value for each x-value, so when the domain and range sets are exchanged, the inverses also have exactly one x-value in the domain for each y-value in the range and, therefore, are a function without having to restrict the domain and range of the original function.*

Review relevant vocabulary as needed before starting the Exit Ticket.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 20: Inverses of Logarithmic and Exponential Functions

Exit Ticket

1. Find the inverse of each function.
   a. $f(x) = \log_2(x) + 2$

   b. $g(x) = e^{x-4}$

   c. $h(x) = 3\log(2 + 3x)$

2. Verify by composition that the given functions are inverses.

   a. $f(x) = 2 - \log(3y + 2)$; $g(x) = \frac{1}{3}(100 \cdot 10^{-x} - 2)$

   b. $f(x) = \ln(x) - \ln(x + 1)$; $g(x) = \frac{e^x}{1-e^x}$

Exit Ticket Sample Solutions

1. **Find the inverse of each function.**

   a. $f(x) = \log_2(x) + 2$

   $$f^{-1}(x) = 2^{x-2}$$

   b. $g(x) = e^{x-4}$

   $$g^{-1}(x) = \ln(x) + 4$$

   c. $h(x) = 3\log(2 + 3x)$

   $$h^{-1}(x) = \frac{1}{3}\left(10^{\frac{x}{3}} - 2\right)$$

2. **Verify by composition that the given functions are inverses.**

   a. $f(x) = 2 - \log(3x + 2);\ g(x) = \frac{1}{3}(100 \cdot 10^{-x} - 2)$

   $$\begin{aligned} f(g(x)) &= f\left(\frac{1}{3}(100 \cdot 10^{-x} - 2)\right) \\ &= 2 - \log\left(3\left(\frac{1}{3}(100 \cdot 10^{-x} - 2)\right) + 2\right) \\ &= 2 - \log(100 \cdot 10^{-x} - 2 + 2) \\ &= 2 - \log(10^{2-x}) \\ &= 2 - (2 - x) \\ &= x \end{aligned}$$

   $$\begin{aligned} g(f(x)) &= \frac{1}{3}\left(100 \cdot 10^{-(2-\log(3x+2))} - 2\right) \\ &= \frac{1}{3}\left(100 \cdot \frac{3x+2}{100} - 2\right) \\ &= \frac{1}{3}(3x + 2 - 2) \\ &= \frac{1}{3}(3x) \\ &= x \end{aligned}$$

b. $f(x) = \ln(x) - \ln(x+1)$; $g(x) = \frac{e^x}{1-e^x}$

$$\begin{aligned} f(g(x)) &= \ln\left(\frac{e^x}{1-e^x}\right) - \ln\left(\frac{e^x}{1-e^x} + 1\right) \\ &= \ln(e^x) - \ln(1-e^x) - \ln\left(\frac{e^x}{1-e^x} + \frac{1-e^x}{1-e^x}\right) \\ &= x - \ln(1-e^x) - \ln\left(\frac{1}{1-e^x}\right) \\ &= x - \ln(1-e^x) - \ln(1) + \ln(1-e^x) \\ &= x \end{aligned}$$

Note that $f(x) = \ln\left(\frac{x}{x+1}\right)$.

$$\begin{aligned} g(f(x)) &= \frac{e^{\ln\left(\frac{x}{x+1}\right)}}{1 - e^{\ln\left(\frac{x}{x+1}\right)}} \\ &= \frac{\frac{x}{x+1}}{1 - \frac{x}{x+1}} \\ &= \frac{\frac{x}{x+1}}{\frac{x+1}{x+1} - \frac{x}{x+1}} \\ &= \frac{\frac{x}{x+1}}{\frac{1}{x+1}} \\ &= \frac{x}{x+1} \cdot \frac{x+1}{1} \\ &= x \end{aligned}$$

Problem Set Sample Solutions

Note that any exponential function's inverse can be described in terms of the natural logarithm or the common logarithm. Depending on how students like to solve exponential functions, any answer involving logarithms can be expressed in a different base.

1. **Find the inverse of each function.**

   a. $f(x) = 3^x$

   $$f^{-1}(x) = \log_3(x)$$

   b. $f(x) = \left(\frac{1}{2}\right)^x$

   $$f^{-1}(x) = \log_{0.5}(x)$$

   Or students may rewrite this as $f(x) = 2^{-x}$ *and then exchange* x *and* y *and solve for* y *getting* $f^{-1}(x) = -\log_2(x)$.

   c. $g(x) = \ln(x-7)$

   $$g^{-1}(x) = e^x + 7$$

d. $h(x) = \frac{\log_3(x+2)}{\log_3(5)}$

By rewriting using the change of base property, $h(x) = \log_5(x+2)$, so

$$h^{-1}(x) = 5^x - 2.$$

e. $f(x) = 3(1.8)^{0.2x} + 3$

$$f^{-1}(x) = 5\log_{(1.8)}\left(\frac{x-3}{3}\right)$$

f. $g(x) = \log_2\left(\sqrt[3]{x-4}\right)$

$$\begin{aligned} g^{-1}(x) &= (2^x)^3 + 4 \\ &= 2^{3x} + 4 \end{aligned}$$

g. $h(x) = \frac{5^x}{5^x+1}$

$$\begin{aligned} x(5^y + 1) &= 5^y \\ x \cdot 5^y + x &= 5^y \\ x \cdot 5^y - 5^y &= -x \\ 5^y(x-1) &= -x \\ 5^y &= -\frac{x}{x-1} \\ h^{-1}(x) &= \log_5\left(-\frac{x}{x-1}\right) \end{aligned}$$

h. $f(x) = 2^{-x+1}$

$$f^{-1}(x) = -\log_2(x) + 1$$

i. $g(x) = \sqrt{\ln(3x)}$

$$g^{-1}(x) = \frac{1}{3}e^{(x^2)}$$

j. $h(x) = e^{\frac{1}{5}x+3} - 4$

$$h^{-1}(x) = 5\ln(x+4) - 15$$

2. Consider the composite function $f \circ g$, composed of invertible functions f and g.

a. Either $f^{-1} \circ g^{-1}$ or $g^{-1} \circ f^{-1}$ is the inverse of the composite function. Which one is it? Explain.

$g^{-1} \circ f^{-1}$ is the inverse of the composite function. In this order, the g and g^{-1} match up, or the f and f^{-1} match up.

b. **Show via composition of functions that your choice of $(f \circ g)^{-1}$ was the correct choice. (Hint: Function composition is associative.)**

$$\begin{aligned}(g^{-1} \circ f^{-1}) \circ (f \circ g) &= g^{-1} \circ (f^{-1} \circ f) \circ g \\ &= g^{-1} \circ g\end{aligned}$$

$$\begin{aligned}(f \circ g) \circ (g^{-1} \circ f^{-1}) &= f \circ (g \circ g^{-1}) \circ f^{-1} \\ &= f \circ f^{-1}\end{aligned}$$

Each time we were able to group the base functions and their inverses together, the result is the identity function. Alternatively, we could have applied the functions to a point x and used the same argument to have a final result of x.

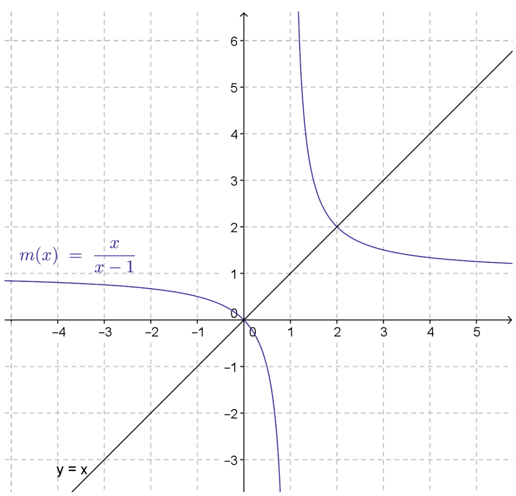
3. **Let $m(x) = \dfrac{x}{x-1}$.**

a. **Find the inverse of m.**

$$\begin{aligned} x &= \frac{y}{y-1} \\ x(y-1) &= y \\ xy - x &= y \\ xy - y &= x \\ y(x-1) &= x \\ y &= \frac{x}{x-1}\end{aligned}$$

The function is its own inverse.

b. **Graph m. How does the graph of m explain why this function is its own inverse?**



Each point on the graph of m reflects to another point on the graph of m when reflected about the line $y = x$.

c. **Think of another function that is its own inverse.**

$f(x) = \dfrac{1}{x}$ has this same reflective property.

Extension:

4. One of the definitions of e involves the infinite series $1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\cdots+\frac{1}{n!}+\cdots$. A generalization exists to define e^x:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots$$

This series definition of e^x allows us to approximate powers of the transcendental number e using strictly rational numbers. This definition is accurate for all real numbers.

a. Verify that the formula given for e can be obtained by substituting 1 for x into the formula for e^x.

$$e^1 = 1 + (1) + \frac{(1)^2}{2} + \frac{(1)^3}{6} + \frac{(1)^4}{24} + \cdots + \frac{(1)^n}{n!} + \cdots$$

b. Use the first seven terms of the series to calculate e, e^2, and e^3.

$$e \approx 1 + 1 + \frac{1^2}{2} + \frac{1^3}{6} + \frac{1^4}{24} + \frac{1^5}{120} + \frac{1^6}{720} = 2.7180\overline{5}$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \frac{2^4}{24} + \frac{2^5}{120} + \frac{2^6}{720} = 6.3\overline{5}$$

$$e^3 \approx 1 + 3 + \frac{3^2}{2} + \frac{3^3}{6} + \frac{3^4}{24} + \frac{3^5}{120} + \frac{3^6}{720} = 17.1625$$

c. Use the inverse of $y = e^x$ to see how accurate your answer to part (b) is.

$$\ln(6.3\overline{5}) \approx 1.849$$

$$\ln(17.1625) \approx 2.843$$

d. Newer calculators and computers use these types of series carried out to as many terms as needed to produce their results for operations that are not otherwise obvious. It may seem cumbersome to calculate these by hand knowing that computers can calculate hundreds and thousands of terms of these series in a single second. Use a calculator or computer to compare how accurate your results from part (b) were to the value given by your technology.

$$e^2 \approx 7.389$$

$$e^3 \approx 20.086$$

We were off by about $1\frac{1}{2}$ and by about 3.

e. $\ln\left(\frac{x}{x-1}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + \cdots + \frac{1}{nx^n} + \cdots$ for $|x| > 1$. What does your response to Exercise 3 part (a) tell you that $\ln(x)$ is equal to?

Since $y = \frac{x}{x+1}$ is its own inverse, you can compose it inside the logarithm to get $\ln(x)$. That means substituting the expression $\frac{x}{x+1}$ for each x in both sides of the equation results in

$$\ln(x) = \frac{1}{\frac{x}{x-1}} + \frac{1}{2\left(\frac{x}{x-1}\right)^2} + \cdots$$

$$= \frac{1}{\frac{x}{x-1}} + \frac{1}{2}\cdot\frac{1}{\left(\frac{x}{x-1}\right)^2} + \frac{1}{3}\cdot\frac{1}{\left(\frac{x}{x-1}\right)^3} + \cdots \frac{1}{n}\cdot\frac{1}{\left(\frac{x}{x-1}\right)^n} + \cdots$$

$$= \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \cdots \frac{1}{n}\left(\frac{x-1}{x}\right)^n + \cdots$$

Lesson 21: Logarithmic and Exponential Problem Solving

Student Outcomes

- Students understand the inverse relationship between logarithms and exponents and apply their understanding to solve real-world problems.

Lesson Notes

This lesson uses the context of radiocarbon dating to understand and apply the inverse relationship between logarithms and exponents when solving problems. A quick Internet search reveals several news stories about recent discoveries of woolly mammoth remains. Radiocarbon dating is one of several methods used by archaeologists and anthropologists to date their findings. Students have studied real-world situations that are modeled by exponential functions since Algebra I. In Algebra II, they were able to develop more precise solutions to modeling and application problems involving exponential functions because they learned to use logarithms to solve equations analytically (**F-BF.B.4**, **F.LE.A.4**). Students also learned about situations that could be modeled by logarithmic functions. In this lesson, students use the inverse relationship between logarithms and exponents as a basis for making sense of and solving real-world problems. Throughout the lesson, students create models, compute using models, and interpret the results (MP.4).

Classwork

Opening (5 minutes)

Have students read the Opening to themselves and jot down one question that they have about the reading. Have them share and discuss this question with a partner. If time permits, students can research the answers to any questions that the teacher or others in the class cannot answer. The following Web-based resources can help the teacher and students learn more about radiocarbon dating and woolly mammoth discoveries.

http://en.wikipedia.org/wiki/Radiocarbon_dating

http://science.howstuffworks.com/environmental/earth/geology/carbon-14.htm

http://en.wikipedia.org/wiki/Woolly_mammoth

http://ngm.nationalgeographic.com/2009/05/mammoths/mueller-text

After a brief whole-class discussion, move students on to the Exploratory Challenge.

Woolly mammoths, elephant-like mammals, have been extinct for thousands of years. In the last decade, several well-preserved woolly mammoths have been discovered in the permafrost and icy regions of Siberia. Using a technique called *radiocarbon (Carbon-14) dating,* scientists have determined that some of these mammoths died nearly $40,000$ years ago.

This technique was introduced in 1949 by the American chemist Willard Libby and is one of the most important tools archaeologists use for dating artifacts that are less than $50,000$ years old. Carbon-14 is a radioactive isotope present in all organic matter. Carbon-14 is absorbed in small amounts by all living things. The ratio of the amount of normal carbon (Carbon-12) to the amount of Carbon-14 in all living organisms remains nearly constant until the organism dies. Then, the Carbon-14 begins to decay because it is radioactive.

Exploratory Challenge/Exercises 1–14 (20 minutes)

Organize students into small groups, and give them about 10 minutes to work these exercises. The goal is for them to create an exponential function to model the data. Monitor groups to make sure they are completing the table entries correctly. There is quite a bit of number sense and quantitative reasoning required in these exercises.

Students may wish to construct the graphs on graph paper. Pay attention to how they scale the graphs. Graphing calculators or other graphing technology can also be used to construct the scatter plots and graphs of the functions.

Exploratory Challenge/Exercises 1–14

By examining the amount of Carbon-14 that remains in an organism after death, one can determine its age. The half-life of Carbon-14 is $5,730$ years, meaning that the amount of Carbon-14 present is reduced by a factor of $\frac{1}{2}$ every $5,730$ years.

1. **Complete the table.**

Years Since Death	0	$5,730$	$11,460$	$17,190$	$22,920$	$28,650$	$34,380$	$40,110$	$45,840$
C-14 Atoms Remaining Per 1.0×10^8 C-12 Atoms	$10,000$	$5,000$	$2,500$	$1,250$	625	312.5	156.25	78.125	38.0625

Let C be the function that represents the number of C-14 atoms remaining per 1.0×10^8 C-12 atoms t years after death.

MP.3

2. **What is $C(11,460)$? What does it mean in this situation?**

   $C(11460) = 2500$
   Each time the years since death increase by $5,730$, you have to reduce the C-14 atoms per 1.0×10^8 C-12 atoms by a factor of $\frac{1}{2}$. After the organism has been dead for $11,460$ years, the organism contains $2,500$ C-14 atoms per 1.0×10^8 C-12 atoms.

3. **Estimate the number of C-14 atoms per 1.0×10^8 C-12 atoms you would expect to remain in an organism that died $10,000$ years ago.**

   It would be slightly more than $2,500$ atoms, so approximately $3,100$.

4. **What is $C^{-1}(625)$? What does it represent in this situation?**

   $C^{-1}(625) = 22,920$
   This represents the number of years since death when there are 625 C-14 atoms per 1.0×10^8 C-12 atoms remaining in the sample.

5. **Suppose the ratio of C-14 to C-12 atoms in a recently discovered woolly mammoth was found to be $0.000\,001$. Estimate how long ago this animal died.**

   We need to write this as a ratio with a denominator equal to 1.0×10^8.

$$0.000\,001 = 1.0\times 10^{-6} = \frac{1.0\times 10^2}{1.0\times 10^8}$$

   So, if the number of C-12 atoms is 1.0×10^8, then the number of C-14 atoms would be 100. This animal would have died between $34,000$ and $40,000$ years ago.

Scaffolding:

Let students use a calculator to do the heavy computational lifting. Students could also model the table information using a spreadsheet. The regression features could be used to generate an exponential model for this situation as well.

6. **Explain why the $C^{-1}(100)$ represents the answer to Exercise 5.**

   $C^{-1}(100)$ means the value of x when $C(x) = 100$. We wanted the time when there would be 100 C-14 atoms for every 1.0×10^8 C-12 atoms.

7. **What type of function best models the data in the table you created in Exercise 1? Explain your reasoning.**

   Since the data is being multiplied by a constant factor each time the years increase by the same amount, this data would be modeled best by an exponential function.

Exercises 8 and 9 provide an opportunity for the teacher to check for student understanding of essential prerequisite skills related to writing and graphing exponential functions. In both Algebra I and Algebra II, students have modeled exponential functions when given a table of values. The challenge for students is to see how well they deal with the half-life parameter. Students may choose to use a calculator to do an exponential regression equation as well. Then, in Exercise 10, the teacher can assess how well students understand that logarithmic and exponential functions are inverses. They create the graph of the inverse of C by exchanging the t and $C(t)$ coordinates and plotting the points. The result of their work in Lesson 20 is that they should now understand that the inverse of any exponential function of the form $y = a \cdot b^{cx}$ is a logarithmic function with base b. Use the questions below to provide additional support as groups are working.

- What type of function makes sense to use to model C?
  - *An exponential function because whenever the time increased by a consistent amount, the value of C was multiplied by $\frac{1}{2}$.*
- How can you justify your choice of function using the graph in Exercise 9?
  - *I can see that every $5{,}730$ years, the amount of C is reduced by a factor of $\frac{1}{2}$. The graph is decreasing but never reaches 0, so it appears to look like a model for exponential decay.*
- What is the base of the function? What is the y-intercept?
  - *The base is $\frac{1}{2}$. The y-intercept would be the amount when $t = 0$.*
- How does the half-life parameter affect how you write the formula for C in Exercise 8?
  - *Since the years are not increasing by 1 but by $5{,}730$, we need to divide the time variable by $5{,}730$ in our exponential function.*
- How do you know that the graph of the function in Exercise 10 is the inverse of the graph of the function in Exercise 9?
  - *Because we exchanged the domain and range values from the table to create the graph of the inverse.*
- What type of function is the inverse of an exponential function?
  - *In the last lesson, we learned that the inverse of an exponential function is a logarithmic function.*

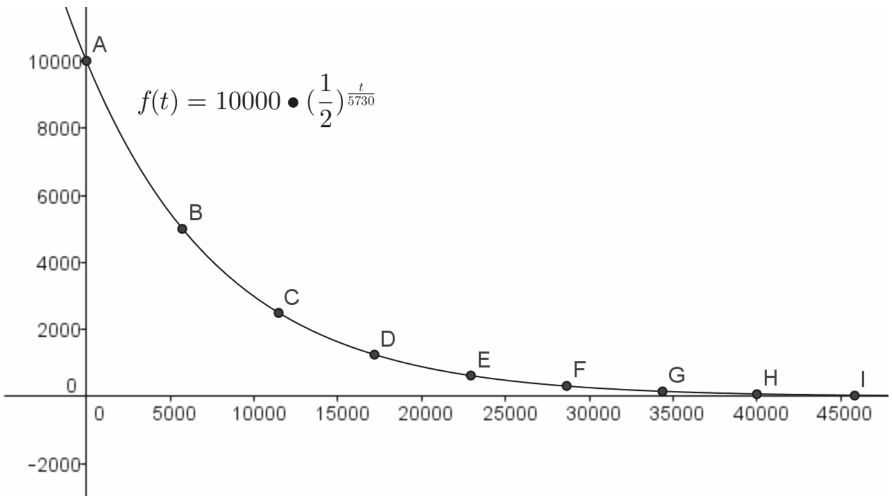
8. Write a formula for C in terms of t. Explain the meaning of any parameters in your formula.

$$C(t) = 10000\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

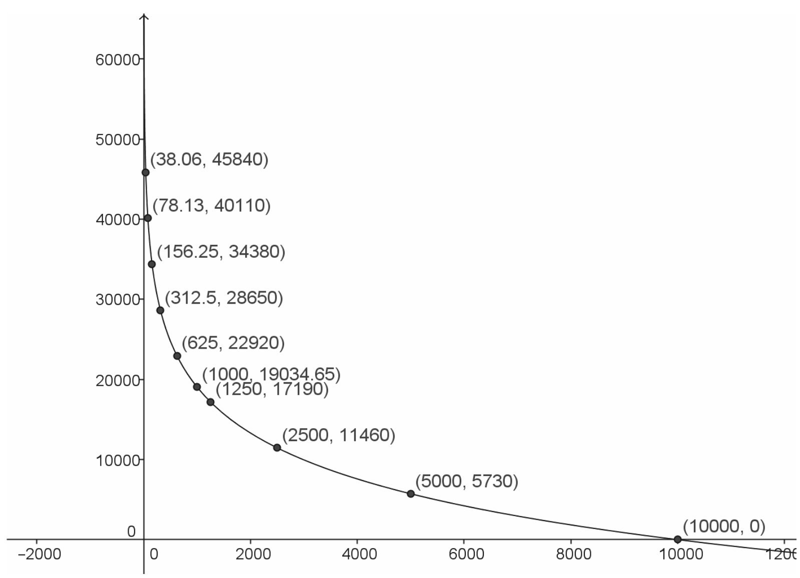
The $10,000$ represents the number of C-14 atoms per 1.0×10^8 C-12 atoms present at the death of the organism.

The $5,730$ and the $\frac{1}{2}$ indicate that the amount is halved every $5,730$ years.

9. Graph the set of points $(t, C(t))$ from the table and the function C to verify that your formula is correct.



10. Graph the set of points $(C(t), t)$ from the table. Draw a smooth curve connecting those points. What type of function would best model this data? Explain your reasoning.



This data would best be modeled by a logarithmic function since the data points represent points on the graph of the inverse of an exponential function.

MP.4

11. **Write a formula that gives the years since death as a function of the amount of C-14 remaining per 1.0×10^8 C-12 atoms.**

Take the equation $C(t) = 10000\left(\frac{1}{2}\right)$, and use the variables x and y in place of t and $C(t)$. Find the inverse by exchanging the x and y variables and solving for y.

$$y = 10000\left(\frac{1}{2}\right)^{\frac{x}{5730}}$$

$$x = 10000\left(\frac{1}{2}\right)^{\frac{y}{5730}}$$

$$\frac{x}{10000} = \left(\frac{1}{2}\right)^{\frac{y}{5730}}$$

$$\log_2\left(\frac{x}{10000}\right) = \log_2\left(\frac{1}{2}\right)^{\frac{y}{5730}}$$

$$\log_2\left(\frac{x}{10000}\right) = \log_2\left(2^{-1}\right)^{\frac{y}{5730}}$$

$$\log_2\left(\frac{x}{10000}\right) = \frac{y}{5730}\log_2\left(2^{-1}\right)$$

$$\log_2\left(\frac{x}{10000}\right) = -\frac{y}{5730}$$

$$y = -5730\log_2\left(\frac{x}{10000}\right)$$

$$f(x) = -5730\log_2\left(\frac{x}{10000}\right)$$

In this formula, x is the number of C-14 atoms for every 1.0×10^8 C-12 atoms, and $f(x)$ is the time since death.

12. **Use the formulas you have created to accurately calculate the following:**

a. **The amount of C-14 atoms per 1.0×10^8 C-12 atoms remaining in a sample after $10,000$ years**

$$y = 10000\left(\frac{1}{2}\right)^{\frac{10000}{5730}}$$

$$C(10000) \approx 2981$$

There are approximately $2,981$ C-14 atoms per 1.0×10^8 C-12 atoms in a sample that died $10,000$ years ago.

b. **The years since death of a sample that contains 100 C-14 atoms per 1.0×10^8 C-12 atoms**

$$f(x) = -5730\, log_2\left(\frac{100}{10000}\right)$$

$$f(100) \approx 38069$$

An organism containing 100 C-14 atoms per 1.0×10^8 C-12 atoms died $38,069$ years ago.

c. $C(25,000)$

$$y = 10000\left(\frac{1}{2}\right)^{\frac{25000}{5730}}$$

$$C(25000) \approx 486$$

After an organism has been dead for $25,000$ years, there are approximately 486 C-14 atoms per 1.0×10^8 C-12 atoms.

d. $C^{-1}(1,000)$

$$f(x) = -5730\log_2\left(\frac{1000}{10000}\right)$$

To find this amount, evaluate $f(1000) \approx 19035$.

When there are $1,000$ C-14 atoms per 1.0×10^8 C-12 atoms, the organism has been dead for $19,035$ years.

13. A baby woolly mammoth that was discovered in 2007 died approximately $39,000$ years ago. How many C-14 atoms per 1.0×10^8 C-12 atoms would have been present in the tissues of this animal when it was discovered?

Evaluate $C(39000) \approx 89$ atoms of C-14 per 10×10^8 atoms of C-12.

14. A recently discovered woolly mammoth sample was found to have a red liquid believed to be blood inside when it was cut out of the ice. Suppose the amount of C-14 in a sample of the creature's blood contained $3,000$ atoms of C-14 per 1.0×10^8 atoms of C-12. How old was this woolly mammoth?

Evaluate $C^{-1}(3000) \approx 9953$. The woolly mammoth died approximately $10,000$ years ago.

Have students present different portions of this Exploratory Challenge, and then lead a short discussion to make sure all students have been able to understand the inverse relationship between logarithmic and exponential functions.

Discuss different approaches to finding the logarithmic function that represented the inverse of C. One of the challenges for students may be dealing with the notation. When applying inverse relationships in real-world situations, be sure to explain the meaning of the variables.

Exercises 15–18 (10 minutes)

In Exercises 15–18, students generalize the work they did in earlier exercises. Because the half-life of C-14 is 5,730 years, the carbon-dating technique only produces valid results for samples up to 50,000 years old. For older samples, scientists can use other radioactive isotopes to date the rock surrounding a fossil and infer the fossil's age from the age of the rock.

Exercises 15–18

Scientists can infer the age of fossils that are older than $50,000$ years by using similar dating techniques with other radioactive isotopes. Scientists use radioactive isotopes with half-lives even longer than Carbon-14 to date the surrounding rock in which the fossil is embedded.

A general formula for the amount A of a radioactive isotope that remains after t years is $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$

where A_0 is the amount of radioactive substance present initially and h is the half-life of the radioactive substance.

15. **Solve this equation for t to find a formula that infers the age of a fossil by dating the age of the surrounding rocks.**

$$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$
$$\frac{A}{A_0} = \left(\frac{1}{2}\right)^{\frac{t}{h}}$$
$$\frac{A}{A_0} = 2^{-\frac{t}{h}}$$
$$\log_2\left(\frac{A}{A_0}\right) = -\frac{t}{h}$$
$$t = -h\log_2\left(\frac{A}{A_0}\right)$$

> *Scaffolding:*
> - For students who need a more concrete approach, select particular values for A_0 and h, and have students solve the equation for t.
> - Do this a few times, and then have students try to generalize the solution steps.
> - Consider also using $A = A_0(2)^{-\frac{t}{h}}$ instead of $\frac{1}{2}$ so that students quickly see the connection to base 2 logarithms.

16. **Let $(x) = A_0\left(\frac{1}{2}\right)^{\frac{x}{h}}$. What is $A^{-1}(x)$?**

$$A^{-1}(x) = -h\,log_2\left(\frac{x}{A_0}\right)$$

17. **Verify that A and A^{-1} are inverses by showing that $A\left(A^{-1}(x)\right) = x$ and $A^{-1}\left(A(x)\right) = x$.**

$$A\left(A^{-1}(x)\right) = A_0\left(\frac{1}{2}\right)^{\frac{-h\log_2\left(\frac{x}{A_0}\right)}{h}}$$
$$= A_0(2^{-1})^{-\log_2\left(\frac{x}{A_0}\right)}$$
$$= A_0(2)^{\log_2\left(\frac{x}{A_0}\right)}$$
$$= A_0\left(\frac{x}{A_0}\right)$$
$$= x$$

And

$$A^{-1}\left(A(x)\right) = -h\log_2\left(\frac{A_0\left(\frac{1}{2}\right)^{\frac{x}{h}}}{A_0}\right)$$
$$= -h\log_2\left(\frac{1}{2}\right)^{\frac{x}{h}}$$
$$= -h\cdot\frac{x}{h}\cdot\log_2\left(\frac{1}{2}\right)$$
$$= -x\cdot -1$$
$$= x$$

18. Explain why, when determining the age of organic materials, archaeologists and anthropologists would prefer to use the logarithmic function to relate the amount of a radioactive isotope present in a sample and the time since its death.

Defining the years since death as a function of the amount of radioactive isotope makes more sense since archaeologists and anthropologists are trying to determine the number of years since the death of an organism from a sample. They know the amount of the radioactive isotope and can use that as an input into the formula to generate the number of years since its death.

Closing (5 minutes)

Ask students to respond to the questions below with a partner or individually in writing.

- Which properties of logarithms and exponents are most helpful when verifying that these types of functions are inverses?
  - *The definition of logarithm, which states that if $x = b^y$, then $y = \log_b(x)$, and the identities, $\log_b(b^x) = x$ and $b^{\log_b(x)} = x$*
- What is the inverse function of $f(x) = 3^x$? What is the inverse of $g(x) = \log(x)$?
  - $f^{-1}(x) = \log_3(x)$, *and* $g^{-1}(x) = 10^x$
- Suppose a function f has a domain that represents time in years and a range that represents the number of bacteria. What would the domain and range of f^{-1} represent?
  - *The domain would be the number of bacteria, and the range would be the time in years.*

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 21: Logarithmic and Exponential Problem Solving

Exit Ticket

Darrin drank a latte with 205 milligrams (mg) of caffeine. Each hour, the caffeine in Darrin's body diminishes by about 8%.

a. Write a formula to model the amount of caffeine remaining in Darrin's system after each hour.

b. Write a formula to model the number of hours since Darrin drank his latte based on the amount of caffeine in Darrin's system.

c. Use your equation in part (b) to find how long it takes for the caffeine in Darrin's system to drop below 50 mg.

Exit Ticket Sample Solutions

Darrin drank a latte with 205 milligrams (mg) of caffeine. Each hour, the caffeine in Darrin's body diminishes by about 8%.

a. Write a formula to model the amount of caffeine remaining in Darrin's system after each hour.

$$c(t) = 205 \cdot (1 - 8\%)^t$$
$$c(t) = 205 \cdot (0.92)^t$$

b. Write a formula to model the number of hours since Darrin drank his latte based on the amount of caffeine in Darrin's system.

$$c = 205(0.92)^t$$
$$\frac{c}{205} = 0.92^t$$
$$\ln\left(\frac{c}{205}\right) = \ln(0.92)^t$$
$$\ln\left(\frac{c}{205}\right) = t \cdot \ln(0.92)$$
$$t = \frac{\ln\left(\frac{c}{205}\right)}{\ln(0.92)}$$

Thus,

$$t(c) = \frac{\ln\left(\frac{c}{205}\right)}{\ln\,(0.92)}.$$

Alternatively,

$$c = 205(0.92)^t$$
$$\frac{c}{205} = 0.92^t$$
$$\log_{0.92}\left(\frac{c}{205}\right) = t.$$

And by the change of base property,

$$t(c) = \frac{\ln\left(\frac{c}{205}\right)}{\ln(0.92)}.$$

c. Use your equation in part (b) to find how long it takes for the caffeine in Darrin's system to drop below 50 mg.

$$t = \frac{\ln\left(\frac{50}{205}\right)}{\ln(0.92)}$$

It would take approximately 16.922 hours for the caffeine to drop to 50 mg; therefore, it would take approximately 17 hours for the caffeine to drop below 50 mg.

Problem Set Sample Solutions

1. A particular bank offers 6% interest per year compounded monthly. Timothy wishes to deposit $\$1,000$.

 a. What is the interest rate per month?

$$\frac{0.06}{12} = 0.005$$

 b. Write a formula for the amount A Timothy has after n months.

$$A = 1000(1.005)^n$$

 c. Write a formula for the number of months it takes Timothy to have A dollars.

$$n = \frac{\ln\left(\frac{A}{1000}\right)}{\ln(1.005)}$$

 d. Doubling-time is the amount of time it takes for an investment to double. What is the doubling-time of Timothy's investment?

$$n = \frac{\ln(2)}{\ln(1.005)}$$
$$\approx 138.98$$

 It takes 139 months for Timothy's investment to double.

 e. In general, what is the doubling-time of an investment with an interest rate of $\frac{r}{12}$ per month?

$$n = \frac{\ln(2)}{\ln\left(1 + \frac{r}{12}\right)}$$

2. A study done from 1950 through 2000 estimated that the world population increased on average by 1.77% each year. In 1950, the world population was $2,519$ million.

 a. Write a formula for the world population t years after 1950. Use p to represent the world population.

$$p = 2519(1.0177)^t$$

 b. Write a formula for the number of years it takes to reach a population of p.

$$t = \frac{\ln\left(\frac{p}{2519}\right)}{\ln(1.0177)}$$

 c. Use your equation in part (b) to find when the model predicts that the world population is 10 billion.

$$t = \frac{\ln\left(\frac{10000}{2519}\right)}{\ln(1.0177)}$$
$$\approx 78.581$$

 According to the model, it takes about $78\frac{1}{2}$ years from 1950 for the world population to reach 10 billion; this would be in 2028.

3. **Consider the case of a bank offering r (given as a decimal) interest per year compounded monthly, if you deposit $\$P$.**

   a. **What is the interest rate per month?**

   $\frac{r}{12}$

   b. **Write a formula for the amount A you have after n months.**

   $A = P\left(1 + \frac{r}{12}\right)^n$

   c. **Write a formula for the number of months it takes to have A dollars.**

$$\ln\left(\frac{A}{P}\right) = n \cdot \ln\left(1 + \frac{r}{12}\right)$$
$$n = \frac{\ln\left(\frac{A}{P}\right)}{\ln\left(1 + \frac{r}{12}\right)}$$

   d. **What is the doubling-time of an investment earning 7% interest per year, compounded monthly? Round up to the next month.**

$$2 = \left(1 + \frac{0.07}{12}\right)^n$$
$$\ln(2) = n \cdot \ln\left(1 + \frac{0.07}{12}\right)$$
$$n = \frac{\ln(2)}{\ln\left(1 + \frac{0.07}{12}\right)} \approx 119.17$$

   It would take 120 months or 10 years in order to double an investment earning 7% interest per year, compounded monthly.

4. **A half-life is the amount of time it takes for a radioactive substance to decay by half. In general, we can use the equation $A = P\left(\frac{1}{2}\right)^t$ for the amount of the substance remaining after t half-lives.**

   a. **What does P represent in this context?**

   The initial amount of the radioactive substance

   b. **If a half-life is 20 hours, rewrite the equation to give the amount after h hours.**

$$t = \frac{h}{20}$$
$$A = P\left(\frac{1}{2}\right)^{\frac{h}{20}}$$

   c. **Use the natural logarithm to express the original equation as having base e.**

   $A = Pe^{\ln\left(\frac{1}{2}\right)t}$

d. The formula you wrote in part (c) is frequently referred to as the "Pert" formula, that is, Pe^{rt}. Analyze the value you have in place for r in part (c). What do you notice? In general, what do you think r represents?

$$r = \ln\left(\frac{1}{2}\right)$$

It seems like the r value always represents the natural logarithm of the growth rate per t. If t is a number of half-lives, then r is the natural logarithm of $\frac{1}{2}$.

e. Jess claims that any exponential function can be written with base e; is she correct? Explain why.

She is correct. No matter what the original rate is, say b, we can always rewrite the rate as $e^{ln(b)}$, so e is always a possible base for an exponential function. Similarly, we could always rewrite logarithms in terms of the natural logarithm.

5. If caffeine reduces by about 10% per hour, how many hours h does it take for the amount of caffeine in a body to reduce by half (round up to the next hour)?

$$\frac{1}{2} = 1 \cdot (0.9)^h$$

$$h = \frac{\ln\left(\frac{1}{2}\right)}{\ln(0.9)}$$

$$\approx 6.5788$$

It takes about 7 hours for the caffeine to reduce by half.

6. Iodine-123 has a half-life of about 13 hours, emits gamma-radiation, and is readily absorbed by the thyroid. Because of these facts, it is regularly used in nuclear imaging.

a. Write a formula that gives you the percent p of iodine-123 left after t half-lives.

$$A = P\left(\frac{1}{2}\right)^t$$

$$\frac{A}{P} = \left(\frac{1}{2}\right)^t$$

$$p = \left(\frac{1}{2}\right)^t$$

b. What is the decay rate per hour of iodine-123? Approximate to the nearest millionth.

$$t = \frac{h}{13}$$

$$p = \left(\frac{1}{2}\right)^{\frac{h}{13}}$$

$$p = \left(\left(\frac{1}{2}\right)^{1/13}\right)^h$$

$$p \approx (0.948\,078)^h$$

Iodine-123 decays by about $0.051\,922$ per hour, or 5.1922%.

c. Use your result to part (b). How many hours h would it take for you to have less than 1% of an initial dose of iodine-123 in your system? Round your answer to the nearest tenth of an hour.

$$\begin{aligned} 0.01 &= (0.948\,078)^h \\ h &= \frac{\ln(0.01)}{\ln(0.948\,078)} \\ &\approx 86.4 \end{aligned}$$

It would take approximately 86.4 hours for you to have less than 1% of an initial does of iodine-123 in your system.

7. An object heated to a temperature of $50°\text{C}$ is placed in a room with a constant temperature of $10°\text{C}$ to cool down. The object's temperature T after t minutes can be given by the function $T(t) = 10 + 40e^{-0.023\,105t}$.

a. How long does it take for the object to cool down to $30°\text{C}$?

$$\begin{aligned} 30 &= 10 + 40e^{-0.023\,105t} \\ \frac{1}{2} &= e^{-0.023\,105t} \\ t &= \frac{\ln\left(\frac{1}{2}\right)}{-0.023\,105} \approx 29.9999 \end{aligned}$$

About 30 minutes

b. Does it take longer for the object to cool from $50°\text{C}$ to $30°\text{C}$ or from $30°\text{C}$ to $10.1°\text{C}$?

Since it is an exponential decay function, it takes longer for the object to cool from $30°\text{C}$ to $10.1°\text{C}$ than it takes for the object to cool from $50°\text{C}$ to $30°\text{C}$. The function levels off as it approaches $10°\text{C}$, so it takes progressively longer. It would take an additional 70 minutes to cool down to $10.1°\text{C}$ after getting to $30°\text{C}$.

c. Will the object ever be $10°\text{C}$ if kept in this room?

It will effectively be $10°\text{C}$ eventually, but mathematically it will never get there. After 400 minutes, the temperature will be about $10.0001°\text{C}$.

d. What is the domain of T^{-1}? What does this represent?

The domain of the inverse represents the possible temperatures that the object could be, so $(10, 50]$.

8. The percent of usage of the word *judgment* in books can be modeled with an exponential decay curve. Let P be the percent as a function of x, and let x be the number of years after 1900; then, $P(x) = 0.022\,046\,5 \cdot e^{-0.007\,994\,1x}$.

a. According to the model, in what year was the usage 0.1% of books?

$$P^{-1}(x) = \frac{\ln\left(\frac{x}{0.022\,046\,5}\right)}{-0.007\,995\,1}$$

According to the inverse of the model, we get a value of -189, which corresponds to the year 1711.

b. When does the usage of the word *judgment* drop below 0.001% of books? This model was made with data from 1950 to 2005. Do you believe your answer is accurate? Explain.

We get a value of 387, which corresponds to the year $2,287$. It is unlikely that the model would hold up well in either part (a) or part (b) because these years are so far in the past and future.

c. **Find P^{-1}. What does the domain represent? What does the range represent?**

$$x = 0.022\,046\,5 \cdot e^{-0.007\,994\,1y}$$
$$-0.0\,07\,994\,1y = \ln\left(\frac{x}{0.022\,046\,5}\right)$$
$$y = \frac{\ln\left(\frac{x}{0.022\,046\,5}\right)}{-0.007\,994\,1}$$

The domain is the percent of books containing the word "judgment" while the range is the number of years after $1,900$.

Name ____ Date ____________________

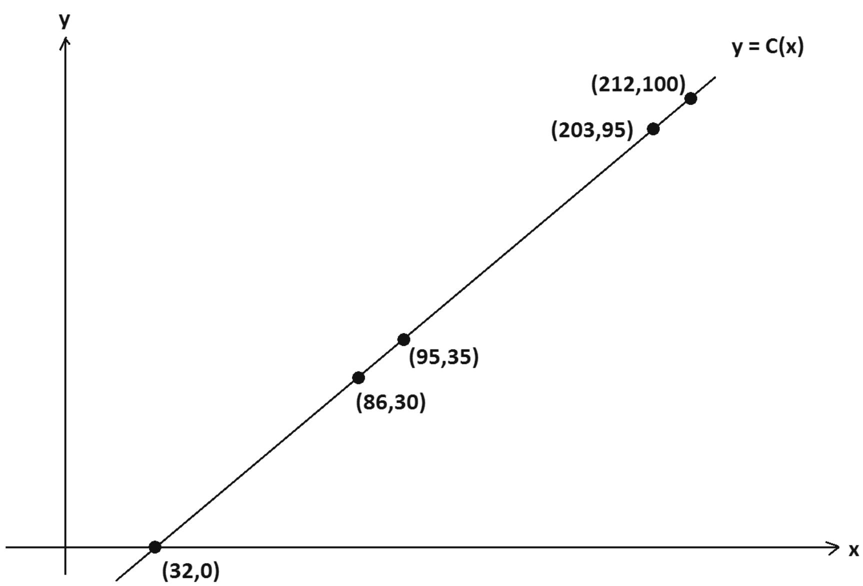
1. Let C be the function that assigns to a temperature given in degrees Fahrenheit its equivalent in degrees Celsius, and let K be the function that assigns to a temperature given in degrees Celsius its equivalent in degrees Kelvin.

   We have $C(x) = \frac{5}{9}(x - 32)$ and $K(x) = x + 273$.

   a. Write an expression for $K\big(C(x)\big)$ and interpret its meaning in terms of temperatures.

   b. The following shows the graph of $y = C(x)$.

      According to the graph, what is the value of $C^{-1}(95)$?



c. Show that $C^{-1}(x) = \frac{9}{5}x + 32$.

A weather balloon rises vertically directly above a station at the North Pole. Its height at time t minutes is $H(t) = 500 - \frac{500}{2^t}$ meters. A gauge on the balloon measures atmospheric temperature in degrees Celsius.

Also, let T be the function that assigns to a value y the temperature, measured in Kelvin, of the atmosphere y meters directly above the North Pole on the day and hour the weather balloon is launched. (Assume that the temperature profile of the atmosphere is stable during the balloon flight.)

d. At a certain time t minutes, $K^{-1}\left(T\left(H(t)\right)\right) = -20$. What is the readout on the temperature gauge on the balloon at this time?

e. Find, to one decimal place, the value of $H^{-1}(300) = -20$, and interpret its meaning.

2. Let f and g be the functions defined by $f(x) = 10^{\frac{x+2}{3}}$ and $g(x) = \log\left(\frac{x^3}{100}\right)$ for all positive real numbers, x. (Here the logarithm is a base-ten logarithm.)

   Verify by composition that f and g are inverse functions to each other.

3. Water from a leaky faucet is dripping into a bucket. Its rate of flow is not steady, but it is always positive. The bucket is large enough to contain all the water that will flow from the faucet over any given hour.

   The table below shows V, the total amount of water in the bucket, measured in cubic centimeters, as a function of time t, measured in minutes, since the bucket was first placed under the faucet.

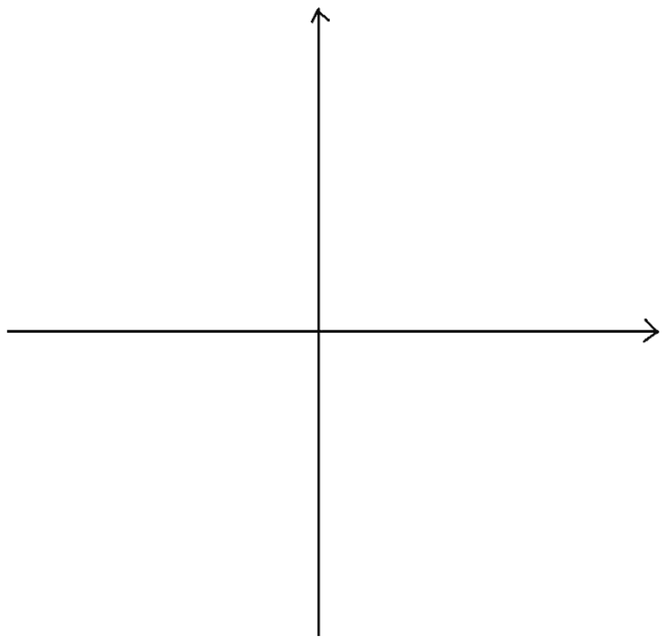
t (minutes)	0	1	2	2.5	3.7	5	10
$V(t)$ (cubic cm)	0	10.2	25.1	32.2	40.4	63.2	69.2

   a. Explain why V is an invertible function.

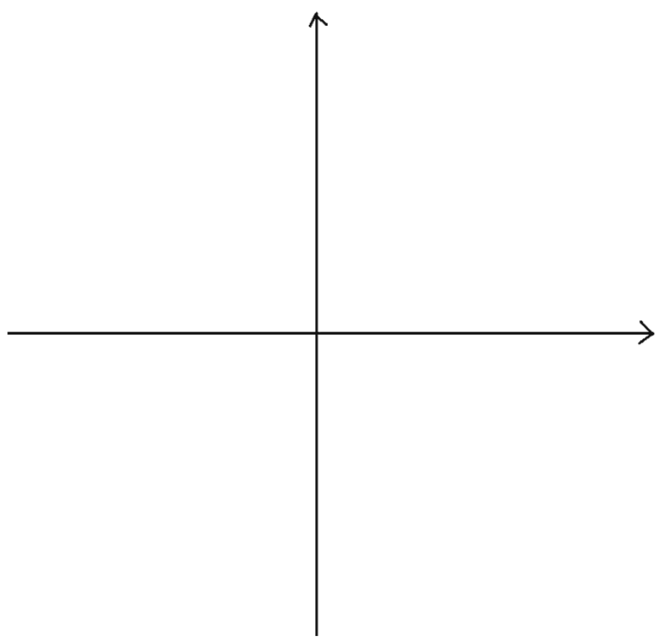
   b. Find $V^{-1}(63.2)$, and interpret its meaning in the context of this situation.

4.

a. Draw a sketch of the graph of $y = \frac{1}{x}$.



b. Sketch the graph of $y = \frac{x}{x-1}$, being sure to indicate its vertical and horizontal asymptotes.



Let f be the function defined by $f(x) = \frac{x}{x-1}$ for all real values x different from 1.

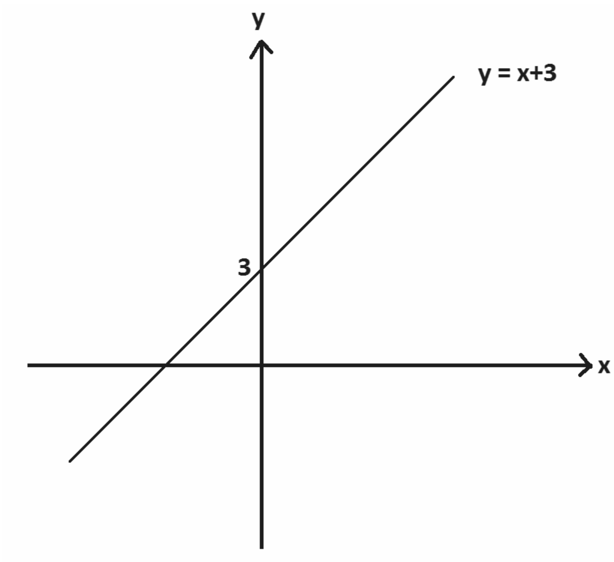
c. Find $f\left(f(x)\right)$ for x, a real number different from 1. What can you conclude about $f^{-1}(x)$?

5. Let f be the function given by $f(x) = x^2 + 3$.

   a. Explain why f is not an invertible function on the domain of all real numbers.

b. Describe a set S of real numbers such that if we restrict the domain of f to S, the function f has an inverse function. Be sure to explain why f has an inverse for your chosen set S.

6. The graph of $y = x + 3$ is shown below.

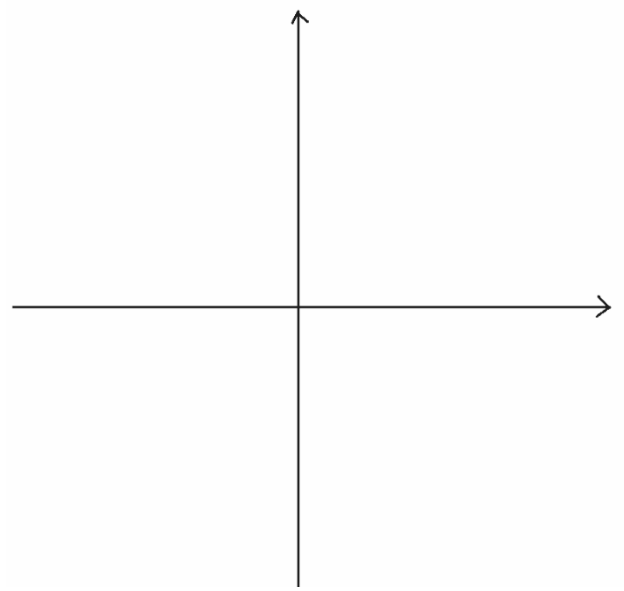


Consider the rational function h given by $h(x) = \frac{x^2-x-12}{x-4}$.

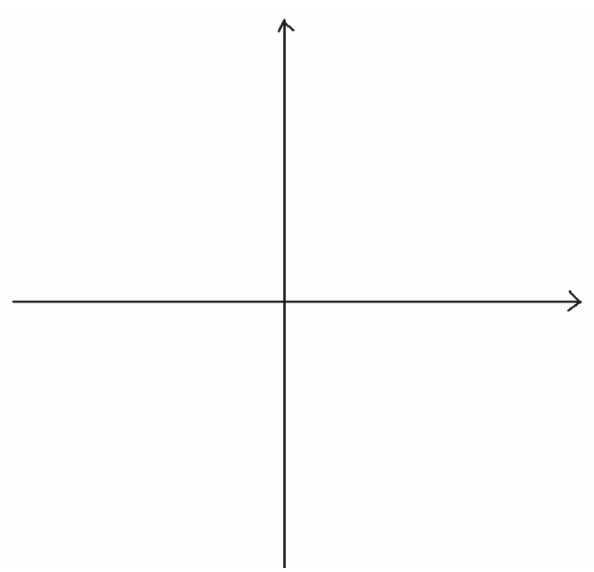
Simon argues that the graph of $y = h(x)$ is identical to the graph of $y = x + 3$. Is Simon correct? If so, how does one reach this conclusion? If not, what is the correct graph of $y = h(x)$? Explain your reasoning throughout.

7. Let f be the function given by $f(x) = 2^x$ for all real values x, and let g be the function given by $g(x) = \log_2(x)$ for positive real values x.

    a. Sketch a graph of $y = f\big(g(x)\big)$. Describe any restrictions on the domain and range of the functions and the composite functions.



    b. Sketch a graph of $y = g\big(f(x)\big)$. Describe any restrictions on the domain and range of the functions and the composite functions.



8. Let f be the rational function given by $f(x) = \frac{x+2}{x-1}$ and g be the rational function given by $g(x) = \frac{x-2}{x+1}$.

   a. Write $f(x) \div g(x)$ as a rational expression.

   b. Write $f(x) + g(x)$ as a rational expression.

   c. Write $f(x) - g(x)$ as a rational expression.

   d. Write $\frac{2f(x)}{f(x)+g(x)}$ as a rational expression.

   e. Ronaldo says that f is the inverse function to g. Is he correct? How do you know?

f. Daphne says that the graph of f and the graph of g each have the same horizontal line as a horizontal asymptote. Is she correct? How do you know?

Let $r(x) = f(x) \cdot g(x)$, and consider the graph of $y = r(x)$.

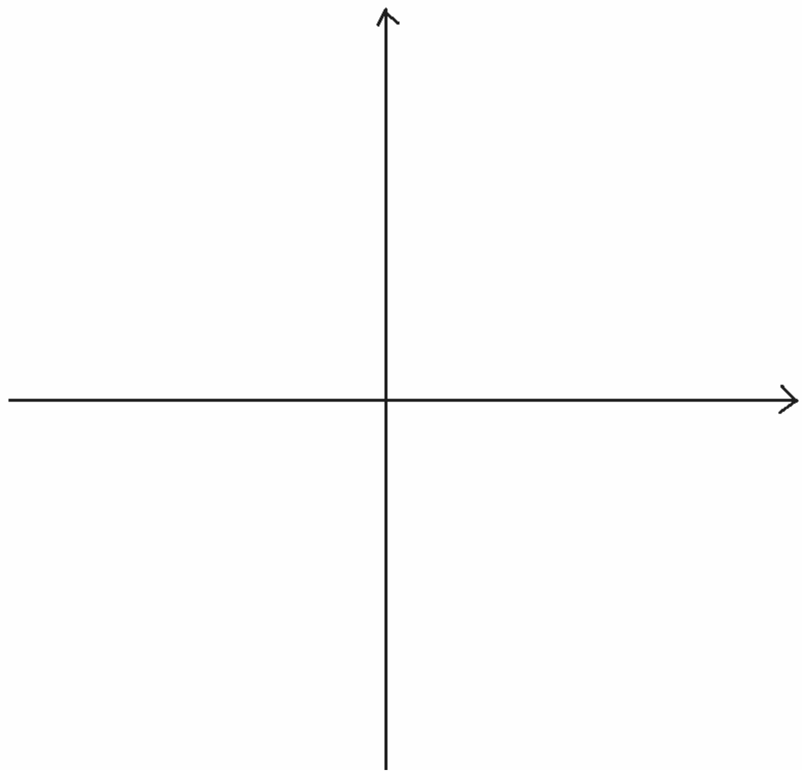
g. What are the x-intercepts of the graph of $y = r(x)$?

h. What is the y-intercept of $y = r(x)$?

i. At which x-values is $r(x)$ undefined?

j. Does the graph of $y = r(x)$ have a horizontal asymptote? Explain your reasoning.

k. Give a sketch of the graph of $y = r(x)$ which shows the broad features you identified in parts (g)–(j).



9. An algae growth in an aquarium triples in mass every two days. The mass of algae was 2.5 grams on June 21, considered *day zero*, and the following table shows the mass of the algae on later days.

d (day number)	0	2	3	4	8	10
mass (grams)	2.5	7.5	13.0	22.5	202.5	607.5

Let $m(d)$ represent the mass of the algae, in grams, on day d. Thus, we are regarding m as a function of time given in units of days. Our time measurements need not remain whole numbers. (We can work with fractions of days too, for example.)

a. Explain why m is an invertible function of time.

b. According to the table, what is the value of $m^{-1}(202.5)$? Interpret its meaning in the context of this situation.

c. Find a formula for the inverse function m, and use your formula to find the value of $m^{-1}(400)$ to one decimal place.

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	**STEP 2** Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	**STEP 3** A correct answer with some evidence of reasoning or application of mathematics to solve the problem, OR an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	**STEP 4** A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	a F-BF.A.1c	Student shows little or no understanding of function composition.	Student attempts to compose functions but makes mathematical mistakes.	Student composes functions correctly but does not interpret the meaning or interprets incorrectly.	Student composes functions and interprets the meaning correctly.
	b F-BF.B.4c	Student shows little or no understanding of inverse functions.	Student finds the y-value or states the value does not exist.	Student attempts to find the x-value but makes a mathematical mistake when finding the value from the graph.	Student finds the correct value from the graph.
	c F-BF.B.4d	Student shows little or no understanding of inverse functions.	Student shows minimal understanding of inverse functions.	Student finds the inverse function but does not solve for y.	Student correctly finds the inverse functions, solving for y.
	d F-BF.A.1c	Student shows little or no understanding of the meaning of function composition.	Student shows minimal understanding of the meaning of function composition.	Student understands the meaning of $T(H(t))$ but does not answer the question correctly.	Student understands the meaning of $T(H(t))$ and answers the question correctly.
	e F-BF.A.1c	Student shows little or no understanding of the inverse of a function.	Student equates $H(t)$ to 300 but does not solve for t.	Student equates $H(t)$ to 300 and solves for t but does not round correctly or does not explain the meaning.	Student equates $H(t)$ to 300 and solves for t, rounding correctly and explaining the meaning.

2	**F-BF.B.4b**	Student shows little or no understanding of inverse functions.	Student attempts to find $f\left(g(x)\right)$ and $g\left(f(x)\right)$, but both are incorrect.	Student attempts to find $f\left(g(x)\right)$ and $g\left(f(x)\right)$, but only one is correct.	Student finds $f\left(g(x)\right)$ and $g\left(f(x)\right)$ correctly, stating that they are inverses.
3	**a** **F-BF.B.4c**	Student shows little or no understanding of inverse functions.	Student attempts to explain why V is invertible, but the explanation is flawed and incomplete.	Student attempts to explain why V is invertible, but the explanation is incomplete.	Student fully and correctly explains why V is invertible.
	b **F-BF.B.4c**	Student shows little or no understanding of inverse functions.	Student equates $V(5)$ to 63.2 but cannot find or explain the meaning.	Student finds $V^{-1}(63.2) = 5$ but does not explain the meaning in this context.	Student finds $V^{-1}(63.2) = 5$ and correctly explains the meaning in this context.
4	**a** **F-IF.C.7d**	Student shows little or no knowledge of graphing rational functions.	Student graphs the function incorrectly, including a point at $x = 0$.	Student graphs the function and has a vertical asymptote at $x = 0$, but the graph is not correct.	Student graphs the function correctly.
	b **F-IF.C.7d**	Student shows little or no knowledge of graphing rational functions.	Student attempts to graph the function but incorrectly includes a point at $x = 0$.	Student graphs the function correctly but does not show the asymptotes.	Student graphs the function correctly showing both asymptotes.
	c **F-BF.B.4b**	Student shows little or no understanding of inverse functions.	Student attempts to find $f\left(f(x)\right)$ but makes major mathematical mistakes.	Student correctly finds $f\left(f(x)\right)$ but does not explain that f is an inverse of itself.	Student correctly finds $f\left(f(x)\right)$ and states that f is an inverse of itself.
5	**a** **F-BF.B.4d**	Student shows little or no understanding of invertible functions.	Student attempts to explain that the function is invertible but does not explain clearly or completely.	Student explains that the function is invertible but does not use a numerical example.	Student explains that the function is invertible using an example of function values.
	b **F-BF.B.4d**	Student shows little or no understanding of invertible functions.	Student incorrectly attempts to find S.	Student shows understanding of invertible functions and restricting the domain but does not explain completely.	Student shows understanding of invertible functions and restricting the domain and explains completely.
6	**F-IF.C.7d** **F-IF.C.9**	Student shows little or no understanding of graphing rational functions.	Student factors the rational expression and gets $x + 3$ but does not explain the discontinuity.	Student factors the rational expression and gets $x + 3$, explains that there is a discontinuity at $x = 4$, but graphs the function incorrectly.	Student factors the rational expression and gets $x + 3$, explains that there is a discontinuity at $x = 4$, and graphs the function correctly.

7	**a** **F-BF.A.1c** **F-BF.B.5**	Student shows little or no understanding of graphing rational functions.	Student graphs the function correctly but does not restrict the domain and range or explain the meaning of $f\big(g(x)\big)$.	Student graphs the function correctly and either restricts the the domain and range correctly or explains the meaning of $f\big(g(x)\big)$.	Student graphs the function correctly, restricts the domain and range correctly, and explains the meaning of $f\big(g(x)\big)$.
	b **F-BF.A.1c** **F-BF.B.5**	Student shows little or no understanding of graphing rational functions.	Student graphs the function correctly but does not restrict the domain and range or explain the meaning of $g\big(f(x)\big)$.	Student graphs the function correctly and either restricts the domain and range correctly or explains the meaning of $g\big(f(x)\big)$.	Student graphs the function correctly, restricts the domain and range, and explains the meaning of $g\big(f(x)\big)$.
8	**a** **A-APR.D.7**	Student shows little or no understanding of rational functions.	Student attempts to find $f(x)/g(x)$ but makes major mathematical errors.	Student attempts to find $f(x)/g(x)$ but makes a minor mathematical error.	Student finds $f(x)/g(x)$ correctly.
	b **A-APR.D.7**	Student shows little or no understanding of rational functions.	Student attempts to find $f(x)+g(x)$ but makes major mathematical errors.	Student attempts to find $f(x)+g(x)$ but makes a minor mathematical error.	Student finds $f(x)+g(x)$ correctly.
	c **A-APR.D.7**	Student shows little or no understanding of rational functions.	Student attempts to find $f(x)-g(x)$ but makes major mathematical errors.	Student attempts to find $f(x)-g(x)$ but makes a minor mathematical error.	Student finds $f(x)-g(x)$ correctly.
	d **A-APR.D.7**	Student shows little or no understanding of rational functions.	Student attempts to find $2f(x)/\big(f(x)+g(x)\big)$ but makes major mathematical errors.	Student attempts to find $2f(x)/\big(f(x)+g(x)\big)$ but makes a minor mathematical error.	Student finds $2f(x)/\big(f(x)+g(x)\big)$ correctly.
	e **F-BF.B.4b**	Student shows little or no understanding of inverse functions.	Student shows minimal understanding of inverse functions and attempts to explain inverse functions but does not use composition.	Student attempts to explain inverse functions through composition but makes a minor error leading to an incorrect answer.	Student correctly explains inverse functions through composition and arrives at the correct answer.
	f **A-APR.D.7** **F-IF.C.7d**	Student shows little or no understanding of horizontal asymptotes.	Student understands that horizontal asymptotes occur at very large or very small values of x but cannot explain the statement.	Student understands that horizontal asymptotes occur at very large or very small values of x but makes a mathematical mistake leading to an incorrect answer.	Student understands that horizontal asymptotes occur at very large or very small values of x and finds the correct asymptote.

	g A-APR.D.7 F-IF.C.7d	Student shows little or no understanding of x-intercepts of rational functions.	Student understands that x-intercepts occur when $y = 0$.	Student finds one x-intercept correctly.	Student finds both x-intercepts correctly.
	h A-APR.D.7 F-IF.C.7d	Student shows little or no understanding of y-intercepts or rational functions.	Student understands that y-intercepts occur when $x = 0$.	Student makes a minor mathematical mistake leading to an incorrect y-intercept.	Student finds the correct y-intercept.
	i A-APR.D.7 F-IF.C.7d	Student shows little or no understanding of the domain of a rational function.	Student understands that a function is undefined when the denominator is zero.	Student understands that a function is undefined when the denominator is zero and finds one x-value where the function is undefined.	Student understands that a function is undefined when the denominator is zero and correctly finds both x-values where the function is undefined.
	j A-APR.D.7 F-IF.C.7d	Student shows little or no understanding of horizontal asymptotes of rational functions.	Student attempts to find a horizontal asymptote but makes major mathematical mistakes.	Student correctly finds the horizontal asymptote but does not explain completely.	Student correctly finds and explains the horizontal asymptote.
	k A-APR.D.7 F-IF.C.7d	Student shows little or no understanding of graphing rational functions.	Student attempts to graph the function, showing the x- and y-intercepts correctly.	Student attempts to graph the function, showing the x- and y-intercepts and horizontal and vertical asymptotes correctly.	Student graphs the function correctly, showing intercepts and asymptotes.
9	a F-BF.B.4d	Student shows little or no understanding of invertible functions.	Student attempts to explain that the function is invertible but makes errors in the explanation.	Student explains that the function is invertible, but the explanation is not complete.	Student explains that the function is invertible using an example completely and correctly.
	b F-BF.B.4c	Student shows little or no understanding of inverse functions.	Student shows some knowledge of invertible functions but does not find $m^{-1}(202.5)$.	Student finds that $m^{-1}(202.5)$ represents the number of days required for the algae to attain a mass of 202.5 kg but calculates the answer incorrectly.	Student finds $m^{-1}(202.5) = 8$ represents the number of days required for the algae to attain a mass of 202.5 kg, which is 8 days.
	c F-BF.B.5	Student shows little or no understanding of inverse functions.	Student attempts to find the inverse function but makes major mathematical mistakes.	Student finds the correct formula but calculates the answer incorrectly.	Student correctly finds the inverse function and calculates the answer, rounding correctly.

Name ____ Date ____________________

1. Let C be the function that assigns to a temperature given in degrees Fahrenheit its equivalent in degrees Celsius, and let K be the function that assigns to a temperature given in degrees Celsius its equivalent in degrees Kelvin.

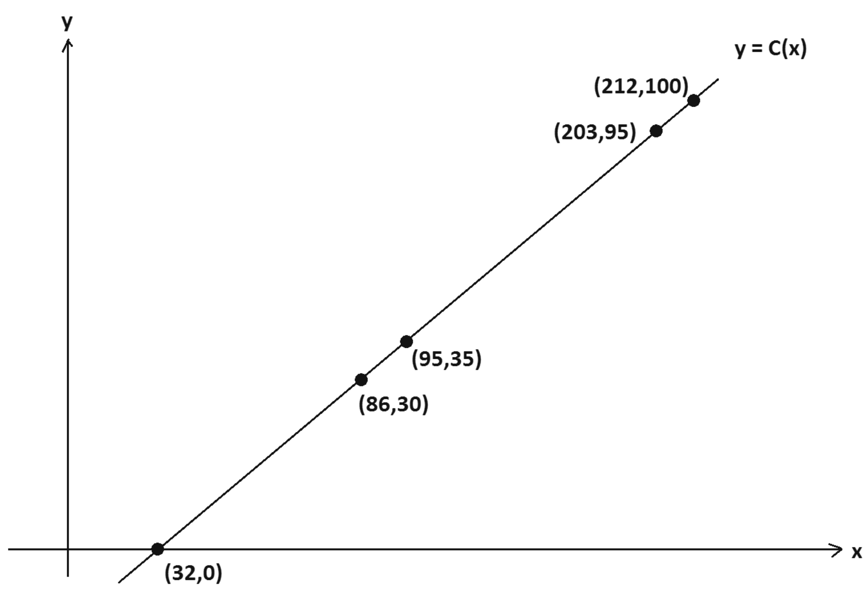
   We have $C(x) = \frac{5}{9}(x - 32)$ and $K(x) = x + 273$.

   a. Write an expression for $K(C(x))$ and interpret its meaning in terms of temperatures.

   $K(C(x)) = C(x) + 273 = \frac{5}{9}(x - 32) + 273$. This shows the arithmetic needed to convert a temperature given in Fahrenheit to its equivalent in Kelvin.

   b. The following shows the graph of $y = C(x)$.

   According to the graph, what is the value of $C^{-1}(95)$?



   We see that $C(203) = 95$; so, $C^{-1}(95) = 203$.

c. Show that $C^{-1}(x) = 32 + (9/5)x$.

If $y = \frac{5}{9}(x-32)$, then $x = \frac{9}{5}y + 32$. This shows that $C^{-1}(x) = \frac{9}{5}x + 32$.

OR

Let d be the function given by $d(x) = \frac{9}{5}x + 32$. Then

$$C(d(x)) = C\left(\frac{9}{5}x + 32\right) = \frac{5}{9}\left(\left(\frac{9}{5}x + 32\right) - 32\right) = \frac{5}{9} \cdot \frac{9}{5}x = x$$

and

$$d(C(x)) = d\left(\frac{5}{9}(x-32)\right) = \frac{9}{5}\left(\frac{5}{9}(x-32)\right) + 32 = (x-32) + 32 = x,$$

which show that d is indeed the inverse function to C.

A weather balloon rises vertically directly above a station at the North Pole. Its height at time t minutes is $H(t) = 500 - \frac{500}{2^t}$ meters. A gauge on the balloon measures atmospheric temperature in degrees Celsius.

Also, let T be the function that assigns to a value y the temperature, measured in Kelvin, of the atmosphere y meters directly above the North Pole on the day and hour the weather balloon is launched. (Assume that the temperature profile of the atmosphere is stable during the balloon flight.)

d. At a certain time t minutes, $K^{-1}\left(T\left(H(t)\right)\right) = -20$. What is the readout on the temperature gauge on the balloon at this time?

$T(H(t))$ is the temperature of the atmosphere, in Kelvin, at height $H(t)$, and $K^{-1}(T(H(t)))$ is this temperature converted to Celsius, which now matches the gauge. So if $K^{-1}\left(T(H(t))\right) = -20$, then the readout on the gauge is -20.

e. Find, to one decimal place, the value of $H^{-1}(300)$ and interpret its meaning.

Now $300 = 500 - \frac{500}{2^t}$ if $\frac{500}{2^t} = 200$, that is $2^t = \frac{5}{2}$. This means $t\log(2) = \log(\frac{5}{2})$; so, $t = \frac{\log(\frac{5}{2})}{\log(2)} \approx 1.3$ minutes. Thus, $H^{-1}(300) \approx 1.3$. This is the time, in minutes, at which the balloon is at a height of 300 meters.

2. Let f and g be the functions defined by $f(x) = 10^{\frac{x+2}{3}}$ and $g(x) = \log\left(\frac{+x^3}{100}\right)$ for all positive real numbers, x. (Here the logarithm is a base-ten logarithm.)

   Verify by composition that f and g are inverse functions to each other.

   Consider $f\big(g(x)\big)$ for a positive real number x. We have

$$f\big(g(x)\big) = 10^{\frac{\log\left(\frac{x^3}{100}\right)+2}{3}} = \left(10^{\log\left(\frac{x^3}{100}\right)}\right)^{\frac{1}{3}} \cdot 10^{\frac{2}{3}}$$

$$= \left(\frac{x^3}{100}\right)^{\frac{1}{3}} \cdot 10^{\frac{2}{3}}$$

$$= \frac{x}{100^{\frac{1}{3}}} \cdot 10^{\frac{2}{3}}$$

$$= \frac{x}{10^{\frac{2}{3}}} \cdot 10^{\frac{2}{3}} = x.$$

   And we also have

$$g\big(f(x)\big) = \log\left(\frac{\left(10^{\frac{x+2}{3}}\right)^3}{100}\right) = \log\left(\frac{10^{x+2}}{100}\right) = \log\left(\frac{100\ 10^x}{100}\right) = \log(10^x) = x.$$

   Thus, f and g are inverse functions to each other.

3. Water from a leaky faucet is dripping into a bucket. Its rate of flow is not steady, but it is always positive. The bucket is large enough to contain all the water that will flow from the faucet over any given hour.

The table below shows V, the total amount of water in the bucket, measured in cubic centimeters, as a function of time t, measured in minutes, since the bucket was first placed under the faucet.

t (minutes)	0	1	2	2.5	3.7	5	10
$V(t)$ (cubic cm)	0	10.2	25.1	32.2	40.4	63.2	69.2

a. Explain why V is an invertible function.

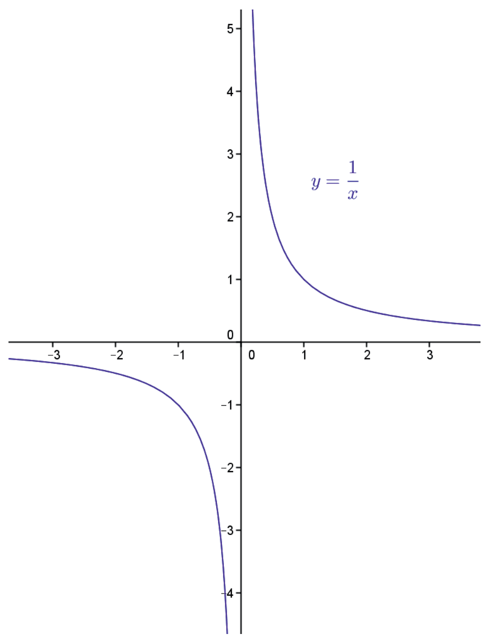
The volume in the bucket is always increasing. So for a given volume (in the range of volumes suitable for this context), there is only one time at which the volume of water in the bucket is that volume. That is, from a given value of the volume, we can determine a unique matching time for that volume. The function V is thus invertible.

b. Find $V^{-1}(63.2)$ and interpret its meaning in the context of this situation.

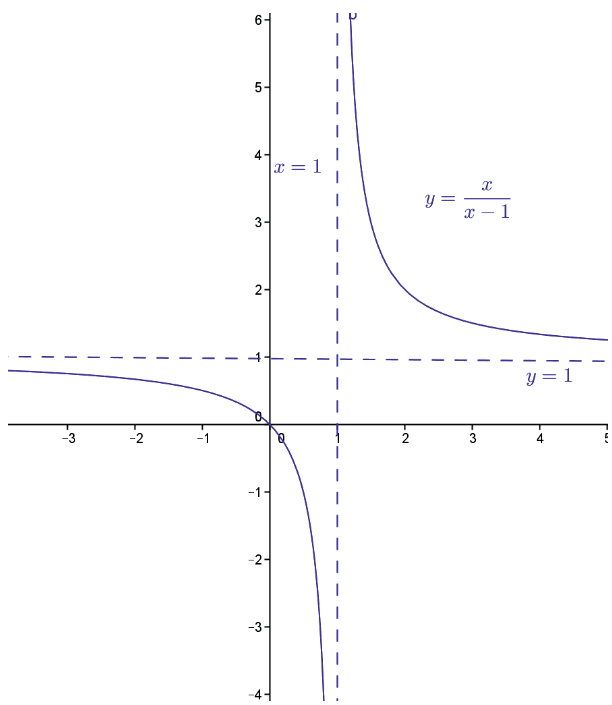
From the table we see that $V(5) = 63.2$; so, $V^{-1}(63.2) = 5$. The time at which there were 63.2 cubic centimeters of water in the bucket was 5 minutes.

4.

a. Draw a sketch of the graph of $y = \frac{1}{x}$.



b. Sketch the graph of $y = \frac{x}{x-1}$, being sure to indicate its vertical and horizontal asymptotes.



Let f be the function defined by $f(x) = \frac{x}{x-1}$ for all real values x different from 1.

c. Find $f\big(f(x)\big)$ for x, a real number different from 1. What can you conclude about $f^{-1}(x)$?

Suppose x is a real number different from 1.

$$f\big(f(x)\big) = f\left(\frac{x}{x-1}\right) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1}$$

$$= \frac{x}{x - 1(x-1)}$$

$$= \frac{x}{x - x + 1}$$

$$= x$$

This shows that the function f itself is the inverse of the function f. We have

$$f^{-1}(x) = f(x) = \frac{x}{x-1}.$$

NOTE: Notice that if x is in the domain of f (i.e., it is a real number different from 1), then f(x) is again a real number different from 1 and so is in the domain of f. We are thus permitted to write f(f(x)).

5. Let f be the function given by $f(x) = x^2 + 3$.

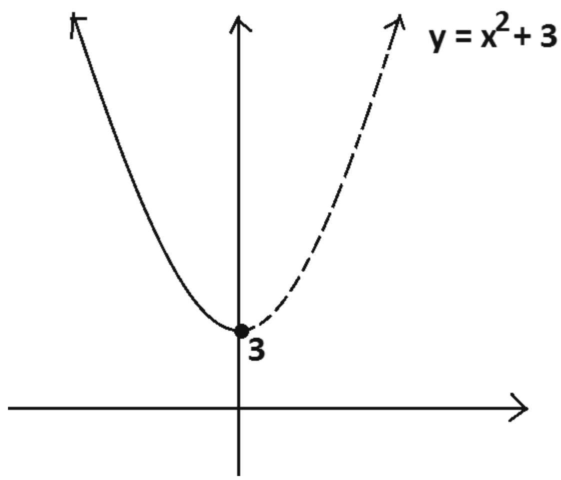
a. Explain why f is not an invertible function on the domain of all real numbers.

$f(-7) = f(7) = 52$, for example, shows that some outputs come from more than one input for this function. The function is not invertible.

b. Describe a set S of real numbers such that if we restrict the domain of f to S, the function f has an inverse function. Be sure to explain why f has an inverse for your chosen set S.

We need to choose a set of real numbers S over which the function is strictly increasing or strictly decreasing.

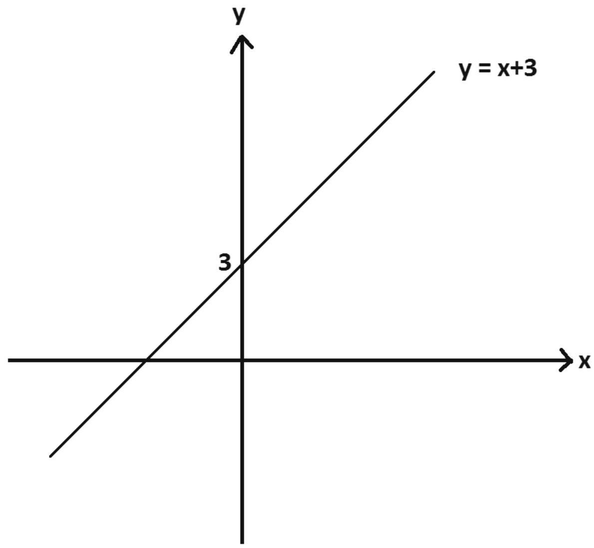
Let's choose S to be the set of all nonpositive real numbers.



Now, the range of f is the set of all real values greater than or equal to 3. If y is a value in this range, we see from the graph that there is only one nonpositive value x such that f(x) = y. Thus, f has an inverse function if we restrict f to the set of nonpositive inputs.

NOTE: Many answers are possible. For example, S could be the set of all positive real numbers, or the set of all real numbers just between −10 and −2, or the set consisting of the number −3 and all the real numbers greater than 3, for example.

6. The graph of $y = x + 3$ is shown below.



Consider the rational function h given by $h(x) = \frac{x^2 - x - 12}{x - 4}$.

Simon argues that the graph of $y = h(x)$ is identical to the graph of $y = x + 3$. Is Simon correct? If so, how does one reach this conclusion? If not, what is the correct graph of $y = h(x)$? Explain your reasoning throughout.

$h(x) = \frac{x^2 - x - 12}{x - 4}$ is defined for all values x different from 4.

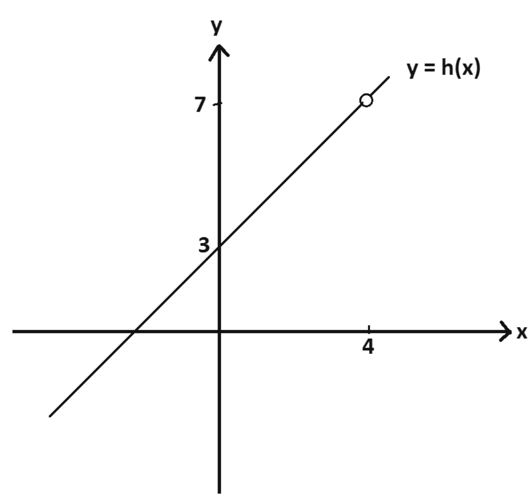
If x is indeed different from 4, we have

$$\frac{x^2 - x - 12}{x - 4} = \frac{(x+3)(x-4)}{x - 4} = x + 3.$$

(Dividing the numerator and denominator each by $x - 4$ is valid as this is a nonzero quantity in the case of $x \neq 4$.)

So we see that $h(x) = \begin{cases} x+3, & \text{if } x \text{ is different from } 4 \\ \text{undefined}, & x = 4 \end{cases}$.

The graph of $y = h(x)$ is shown on the right.

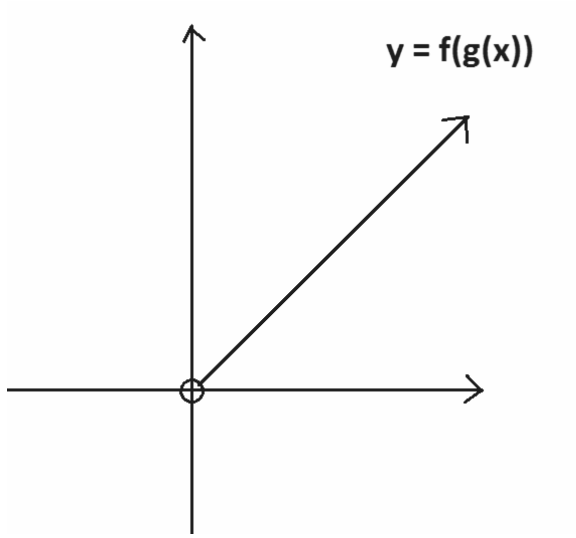


7. Let f be the function given by $f(x) = 2^x$ for all real values x, and let g be the function given by $g(x) = \log_2(x)$ for positive real values x.

   a. Sketch a graph of $y = f\big(g(x)\big)$. Describe any restrictions on the domain and range of the functions and the composite functions.

   $f\big(g(x)\big)$ is meaningful if x is an appropriate input for g, that is, a positive real number, and its output, $g(x)$, is an appropriate input for f (which it shall be, as all real values are appropriate inputs for f). Thus, $f \circ g$ is defined only for positive real inputs.

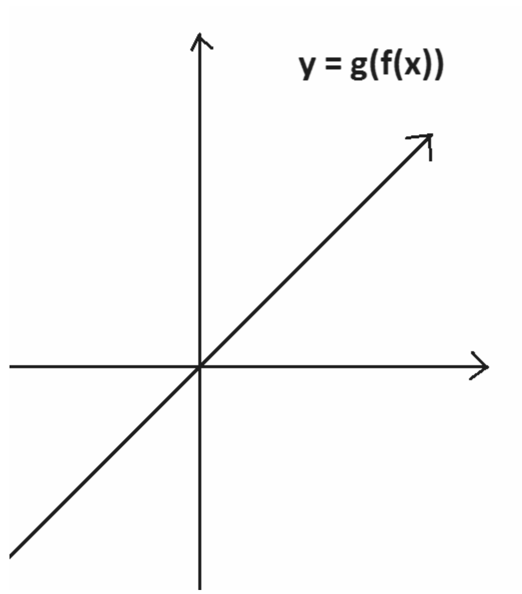
   If x is a positive real number, then $f\big(g(x)\big) = 2^{\log_2 x} = x$, as exponential functions and logarithmic functions (with matching bases) are inverse functions. The graph of $y = f(g(x))$ is shown on the right.



   b. Sketch a graph of $y = g\big(f(x)\big)$. Describe any restrictions on the domain and range of the functions and the composite functions.

   $g(f(x))$ is meaningful if x is an appropriate input for f, which is the case for all real values x, and its output, $f(x)$, is an appropriate input for g, which it shall be as 2^x is a positive value. Thus, $g \circ f$ is defined only for all real inputs.

   If x is a real number, then $g(f(x)) = \log_x(2^x) = x$ as exponential functions and logarithmic functions (with matching bases) are inverse functions. Thus, the graph of $y = g(f(x))$ is as shown on the right.



8. Let f be the rational function given by $f(x) = \frac{x+2}{x-1}$ and g the rational function given by $g(x) = \frac{x-2}{x+1}$.

a. Write $f(x) \div g(x)$ as a rational expression.

We have

$$f(x) \div g(x) = \left(\frac{x+2}{x-1}\right) \div \left(\frac{x-2}{x+1}\right) = \frac{(x+2)(x+1)}{(x-1)(x-2)}.$$

b. Write $f(x) + g(x)$ as a rational expression.

We have

$$f(x) + g(x) = \frac{x+2}{x-1} + \frac{x-2}{x+1} = \frac{(x+2)(x+1) + (x-2)(x-1)}{(x-1)(x+1)} = \frac{2x^2+4}{x^2-1}.$$

c. Write $f(x) - g(x)$ as a rational expression.

We have

$$f(x) - g(x) = \frac{x+2}{x-1} - \frac{x-2}{x+1} = \frac{(x+2)(x+1) - \left(x-2\right)\left(x-1\right)}{\left(x-1\right)(x+1)} = \frac{6x}{x^2-1}.$$

d. Write $\frac{2f(x)}{f(x)+g(x)}$ as a rational expression.

We have $\frac{2f(x)}{f(x)+g(x)} = 2\left(\frac{x+2}{x-1}\right) \div \left(\frac{2x^2+4}{x^2-1}\right) = \frac{2(x+2)(x^2-1)}{(x-1)(2x^2+4)} = \frac{2(x+2)(x-1)(x+1)}{(x-1)\cdot 2(x^2+2)}$

$$= \frac{(x+2)(x+1)}{x^2+2}.$$

e. Ronaldo says that f is the inverse function to g. Is he correct? How do you know?

We have, for example, $f(2) = \frac{4}{1} = 4$ but $g(4) = \frac{2}{3}$. That is, $g\left(f(2)\right)$ is not 2. If f were the inverse function to g, then we should see $g\left(f(2)\right) = 2$. Ronaldo is not correct.

f. Daphne says that the graph of f and the graph of g each have the same horizontal line as a horizontal asymptote. Is she correct? How do you know?

We have $f(x) = \frac{x+2}{x-1} = \frac{x\left(1+\frac{2}{x}\right)}{x\left(1-\frac{1}{x}\right)}$. *If x is a real number large in magnitude, then* $\frac{2}{x}$ *and* $\frac{1}{x}$ *each have values close to zero, and, in this case,* $f(x) \approx \frac{1+0}{1-0} = 1$ *shows that the graph of f has the horizontal line $y = 1$ as an asymptote.*

The same work shows that $g(x) = \frac{x\left(1-\frac{2}{x}\right)}{x\left(1+\frac{1}{x}\right)}$ *also has the line $y = 1$ as a horizontal asymptote.*

Thus, Daphne is indeed correct.

Let $r(x) = f(x) \cdot g(x)$, and consider the graph of $y = r(x)$.

g. What are the x-intercepts of the graph of $y = r(x)$?

For $y = r(x) = \frac{(x+2)(x-2)}{(x+1)(x-1)}$, *we have $y = 0$ when $x = -2$ and $x = 2$. The x-intercepts occur at the points $(-2, 0)$ and $(2, 0)$.*

h. What is the y-intercept of $y = r(x)$?

$r(0) = \frac{(2)(-2)}{(1)(-1)} = 4$. *The y-intercept occurs at the point $(0, 4)$.*

i. At which x-values is $r(x)$ undefined?

$r(x)$ is undefined at $x = 1$ and at $x = -1$.

j. Does the graph of $y = r(x)$ have a horizontal asymptote? Explain your reasoning.

We have

$$y = r(x) = \frac{x^2 - 4}{x^2 - 1} = \frac{x^2\left(1 - \frac{4}{x^2}\right)}{x^2\left(1 - \frac{1}{x^2}\right)}.$$

If x is a real positive number or a real negative number large in magnitude (and so certainly not zero), then

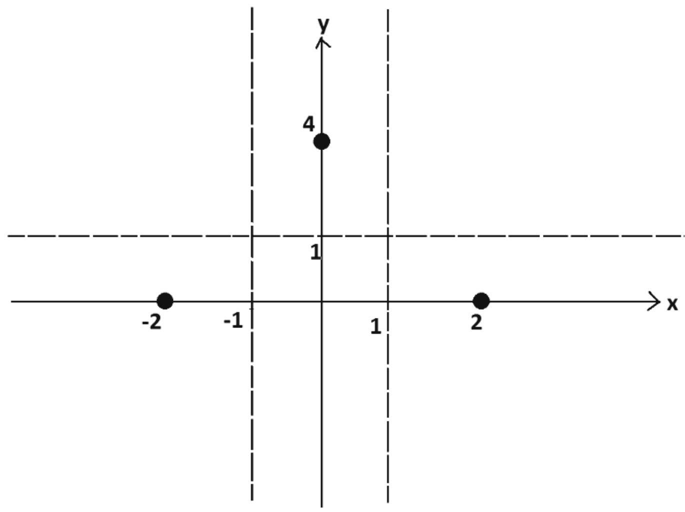
$$r(x) = \frac{1 - \frac{4}{x^2}}{1 - \frac{1}{x^2}} \approx \frac{1 - 0}{1 - 0} = 1.$$

This shows that the graph of $y = r(x)$ closely matches the horizontal line $y = 1$ for large positive and large negative inputs. We have a horizontal asymptote, the line $y = 1$.

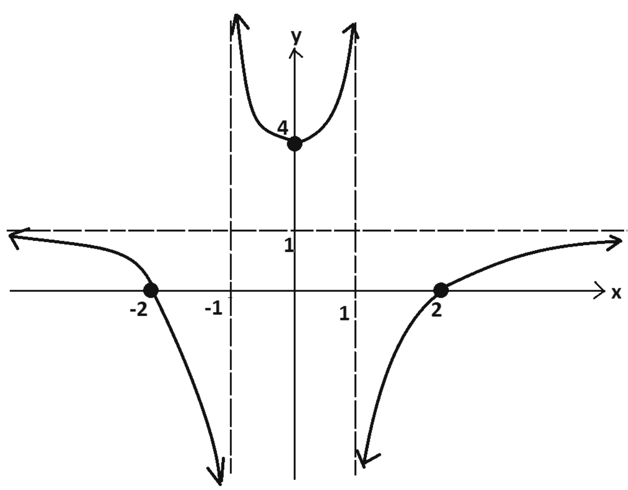
Notice that for values of x large in magnitude, $x^2 - 1$ and $x^2 - 4$ are positive, and $\frac{x^2 - 4}{x^2 - 1} < \frac{x^2 - 1}{x^2 - 1} < 1$, so the graph of $y = r(x)$ always lies below its horizontal asymptote in its long-term behavior.

k. Give a sketch of the graph of $y = r(x)$ which shows the broad features you identified in parts (g)–(j).

The broad features of the graph are indicated as so, with the dashed lines representing asymptotes:



Given the graph of $y = r(x)$ crosses the x-axis only at $x = -2$ and $x = 2$ and the y-axis only at $y = 4$, we deduce that the graph of the curve can only be of the form:



9. An algae growth in an aquarium triples in mass every two days. The mass of algae was 2.5 grams on June 21, considered *day zero*, and the following table shows the mass of the algae on later days.

d (day number)	**0**	**2**	**3**	**4**	**8**	**10**
mass (grams)	**2.5**	**7.5**	**13.0**	**22.5**	**202.5**	**607.5**

Let $m(d)$ represent the mass of the algae, in grams, on day d. Thus, we are regarding m as a function of time given in units of days. Our time measurements need not remain whole numbers. (We can work with fractions of days too, for example.)

a. Explain why m is an invertible function of time.

The algae grows in mass over time, so its mass is an increasing function of time. For each possible value of mass (output), there is only *one* possible time (input) at which the growth has that mass. Thus, we can define an inverse function.

b. According to the table, what is the value of $m^{-1}(202.5)$? Interpret its meaning in the context of this situation.

$m^{-1}(202.5)$ represents the number of days required for the algae to attain a mass of 202.5 grams. According to the table, we have $m^{-1}(202.5) = 8$ days.

c. Find a formula for the inverse function m, and use your formula to find the value of $m^{-1}(400)$ to one decimal place.

The growth is exponential. The initial mass is 2.5 grams, and the mass triples every two days. We see that $m(d) = 2.5 \cdot 3^{\frac{d}{2}}$. To find the inverse function to m, notice that

If $y = 2.5 \cdot 3^{\frac{d}{2}}$, then

$$\log(y) = \log(2.5) + \frac{d}{2}\log(3)$$

$$\log(y) - \log(2.5) = \frac{d}{2}\log(3)$$

(using base-ten logarithms) giving

$$d = \frac{2}{\log(3)}(\log(y) - \log(2.5)) = \frac{2\log\left(\frac{y}{2.5}\right)}{\log(3)}.$$

This shows

$$m^{-1}(y) = \frac{2\log\left(\frac{y}{2.5}\right)}{\log(3)}.$$

To finish, we see

$$m^{-1}(400) = \frac{2\log\left(\frac{400}{2.5}\right)}{\log(3)} \approx 9.2 \text{ days}.$$