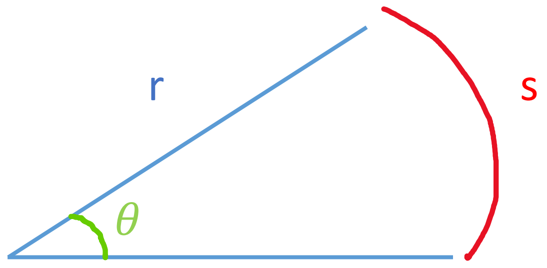
The ABC's of Trigonometry

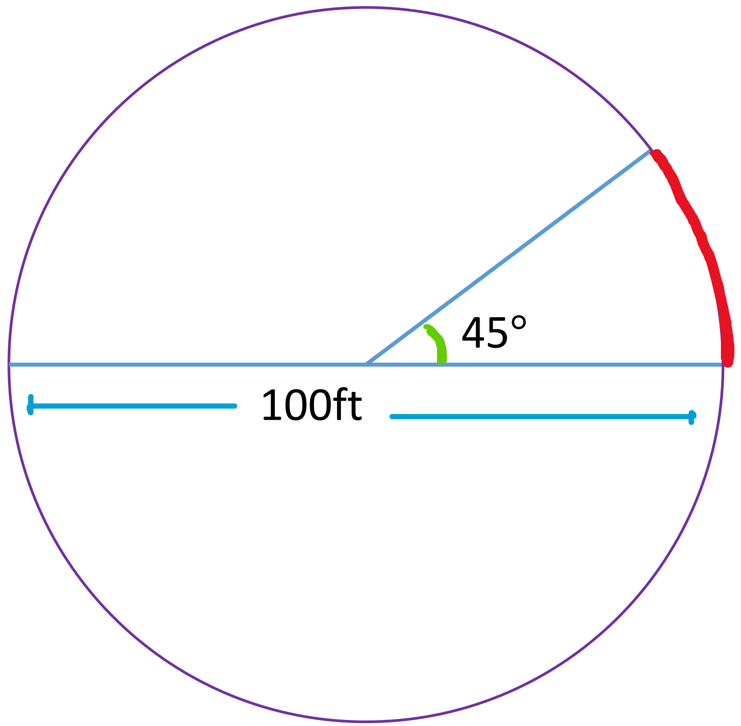
By: The kid who recommends doing your homework

A is for Arc Length

Formula for length of intercepted arc: $s = r\theta$

θ is central angle in radians





*Not to scale

Ex: Susan walked part way around a pond that measures 100ft across before stopping to take a break. She stopped 45° from walking all the way around the pond. How many more feet does Susan need to walk before reaching the start again?

The radius is $\frac{100}{2}$ ft

$50 \times \frac{\pi}{4} = 39.3$

Susan has 39.3ft left to walk.

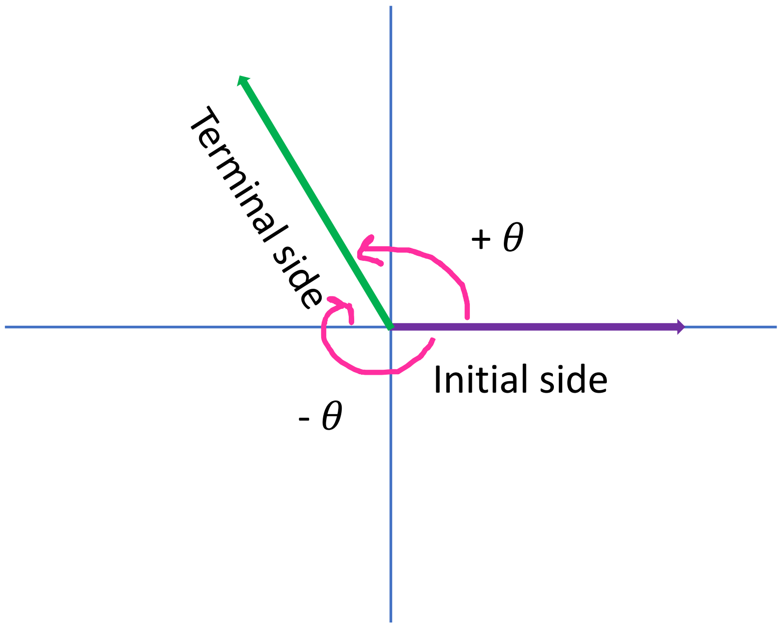
B is for ...

Well what do we need B for?

Moving on.

C is for Coterminal Angles

Angles with the same initial and terminal side are coterminal angles.
In standard position the vertex is at the origin and the initial side of the angle is along the positive x-axis.



Angles are coterminal when they differ by a multiple 360° or 2π rad

Ex: Find the coterminal angles of the following

30°, -150°, $\frac{2}{3}\pi$

$30° + 360° = 390°$
$30° - 720° = -330°$ *720 is 2(360)
$-150° + 360° = 210°$
$-150° - 360° = 510°$
$\frac{2}{3}\pi + 2\pi = \frac{8}{3}\pi$
$\frac{2}{3}\pi - 2\pi = -\frac{4}{3}\pi$

D is for DeMoivre's Theorem: Raising Complex Numbers to a Power

Formula: $z^n = [r\,(\cos\theta + i\sin\theta)]^n$

This means above is = $r^n(\cos n\theta + i\sin n\theta)$

Ex: $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^3$

*θs will always be the same in trigonometric form

In this case r = 1 so

$1^3\left[\cos\left(3\times\frac{\pi}{4}\right) + i\sin\left(3\times\frac{\pi}{4}\right)\right]$ =

$\_\left(\cos\frac{3}{4}\pi + i\ \sin\frac{3}{4}\pi\right)$

* One is omitted because it is understood when there is no coefficient

Ex: $\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^2$

In this case r = 2 so

$\left[2^2\left(\cos\left(2\times\frac{\pi}{3}\right) + i\sin\left(2\times\frac{\pi}{3}\right)\right)\right]$ =

$4\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)$

E is for Ellipse

Vertex

Semi-major axis

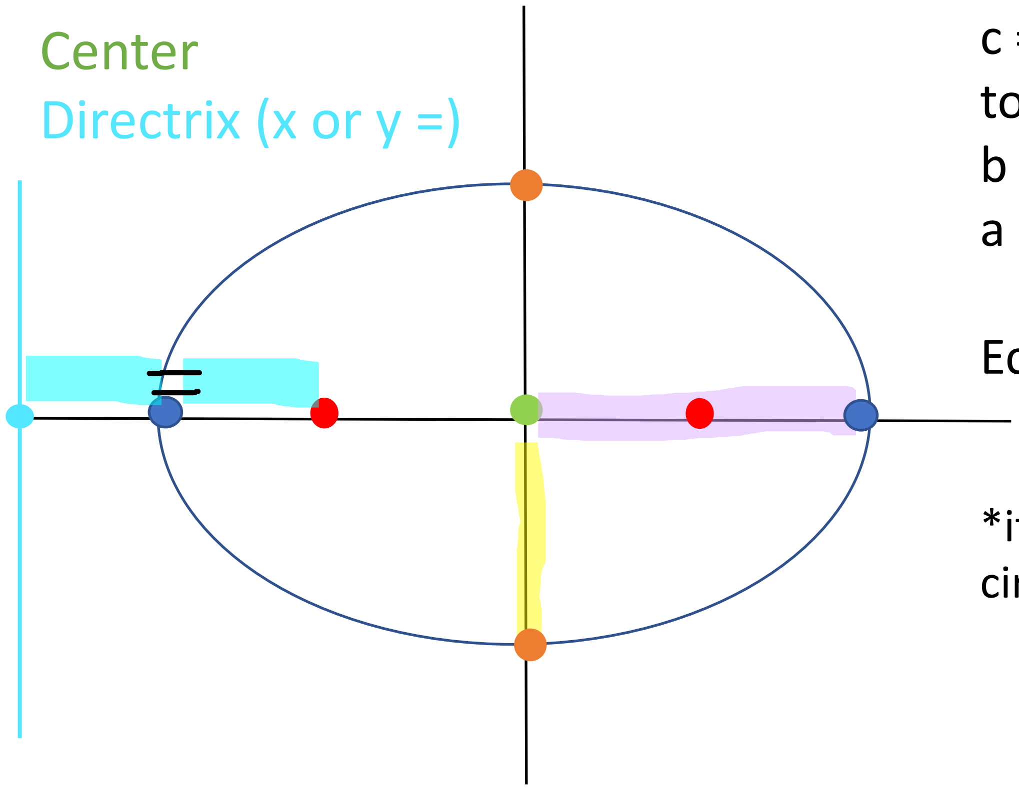
Semi-minor axis

Foci

Co-vertices

Center

Directrix (x or y =)



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

*both ellipse always positive

Pythagorean relation:

$$a^2 = b^2 + c^2$$

c = distance from center to focus

b = semi-minor axis

a = semi-major axis

Eccentricity:

$$e = \frac{c}{a}$$

*if e is equal to 1, it is a circle

Ex: $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Center: (0,0)

nothing + or — from x and y

Horizontal ellipse

> number (4) under x

Semi-major = 2

$\sqrt{a} = \sqrt{4}$

Semi-minor = $\sqrt{3}$

$\sqrt{b} = \sqrt{3}$

C = 1

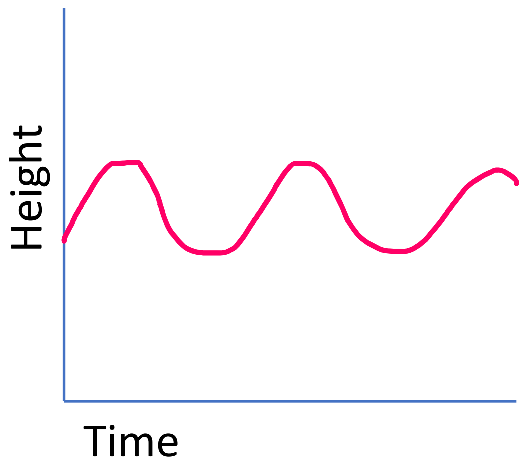
rearrange Pythagorean E. and solve for c

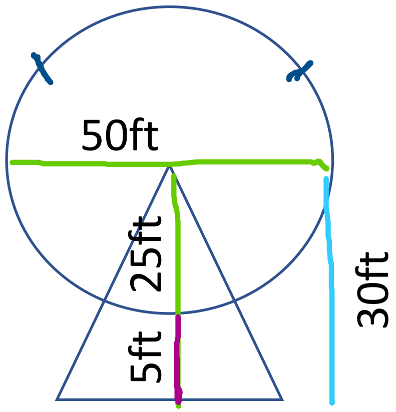
Foci: $(0 \pm 1, 0)$ (1, 0) and (-1, 0)

Vertices: $(0 \pm 2, 0)$ (2, 0) and (-2, 0)

F is for Ferris Wheel Problem

The height of an object moving in a circular path can be modeled by a sinusoid.





Ex: A Ferris wheel 50ft in diameter makes a revolution every 40 seconds. If the center of the wheel is 30ft above the ground, how long after reaching the low point is the rider 50ft above the ground?

f(x) = $a\cos(bx + c) + d$

Period: $\frac{2\pi}{|b|}$ (a full cycle)

A = amplitude

C= phase shift $\frac{-c}{b}$

D= vertical shift

1 rev = 40s

1 period = $\frac{2\pi}{40} = \frac{\pi}{20}$

A = 25ft, b= $\frac{\pi}{20}$, c = 0, d = 30

$$y = -25\cos\frac{\pi}{20}x + 30$$

Using your equation solver you get:

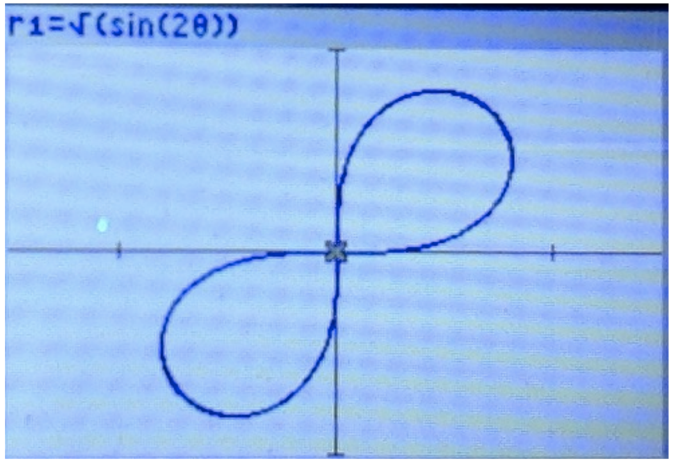
At 15.90 seconds, the rider will reach 50ft

G is for Graphs of Special Curves

Leminscate

$r^2 = a^2 \sin 2\theta$
$r^2 = a^2 \cos 2\theta$

Ex:
$r^2 = \sin 2\theta, 0 \leq \theta \leq 2\pi$



D: $\left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3}{2}\pi\right]$
R: [0, 1]
Continuous on domain
Symmetric about origin
Bounded
Max r: 1
No asymptotes

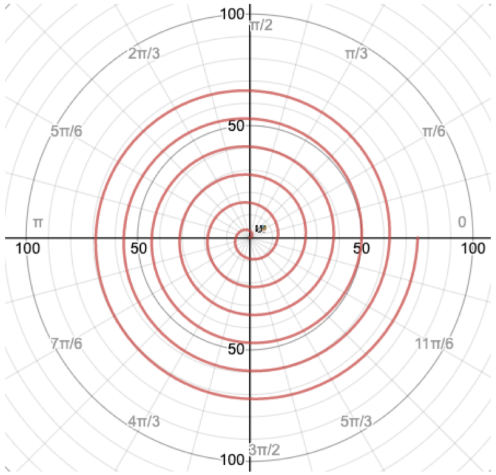
*must use square root to graph

Spiral of Archiedes

$r = \theta$

D:$(-\infty, \infty)$
R:$(-\infty, \infty)$

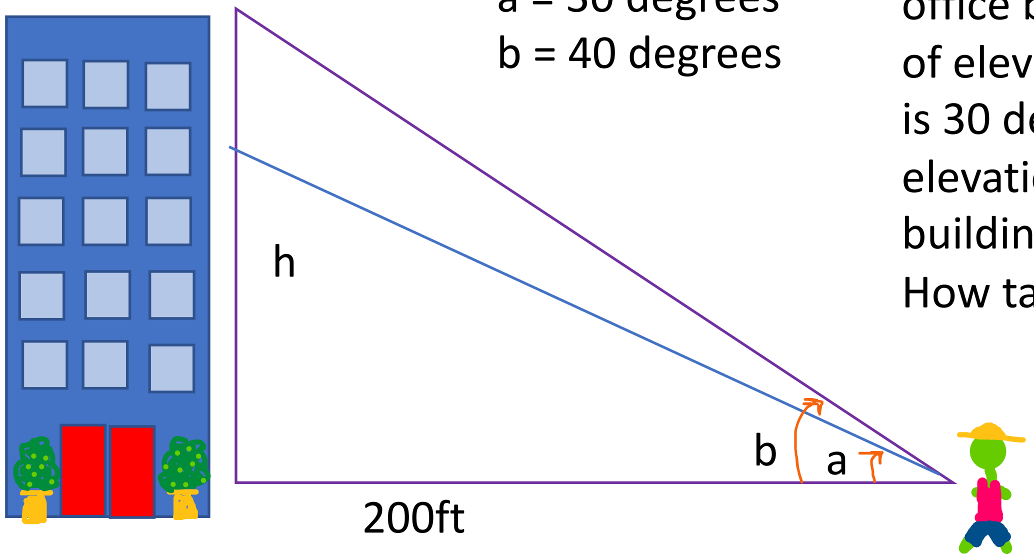
Continuous
Unbounded
No maximum r value
No symmetry
No asymptotes



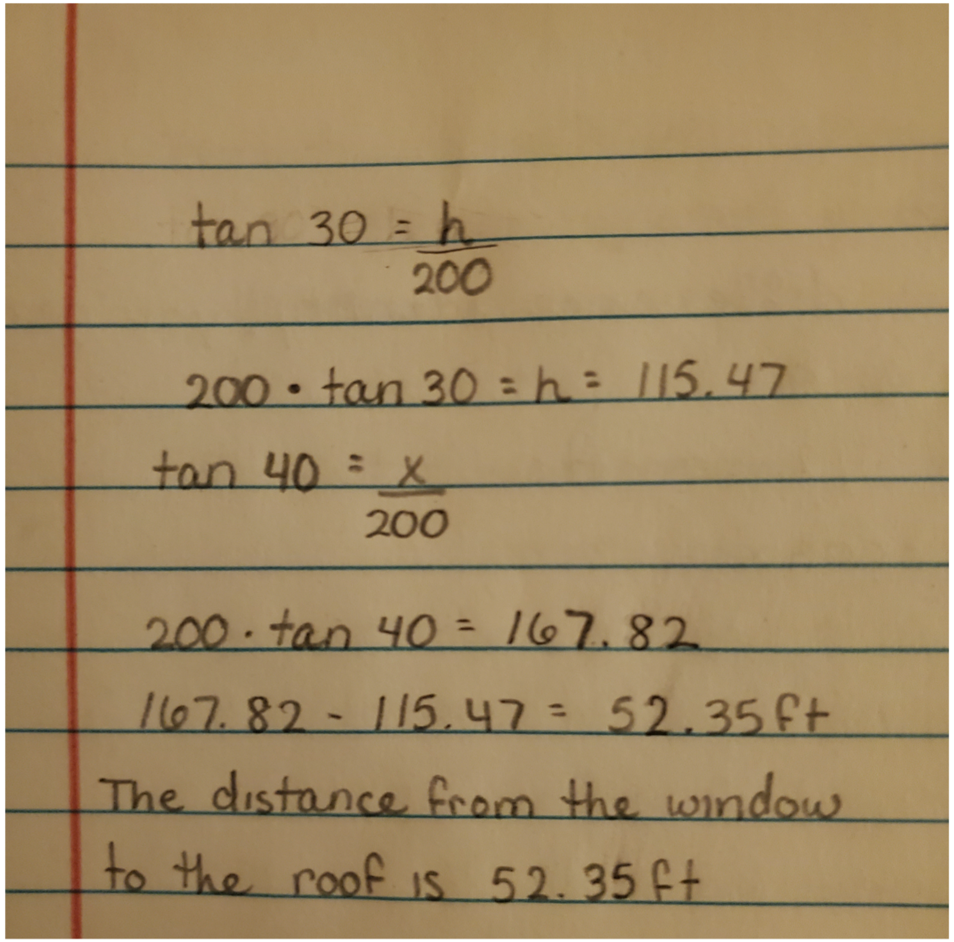
H is for Height Above Ground

Using angles of depression and elevation along with the basic trig functions, you can find something's height

a = 30 degrees
b = 40 degrees



Bob is standing 200ft away from the door to his office building. His angle of elevation to a window is 30 degrees. His angle elevation to the top of the building is 40 degrees. How tall is the building?



$\tan 30 = \frac{h}{200}$

$200 \cdot \tan 30 = h = 115.47$

$\tan 40 = \frac{x}{200}$

$200 \cdot \tan 40 = 167.82$

$167.82 - 115.47 = 52.35$ ft

The distance from the window to the roof is 52.35 ft

I is for Sum and Difference Identities

Cosine Sum or Difference Identity

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

Sine Sum or Difference Identity

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

Prove:

$\cos(\theta + \pi/2) = -\sin\theta$

Recognizing the cos sum identity, you get:

$\cos x . \cos 90° - \sin x \sin 90$

solve the sin + cos w/ known θ

$\cos x \cdot 0 - \sin x \cdot 1 = -\sin\theta$

$0 - \sin x$

$-\sin x = -\sin x$

Prove:

$\sin(x - 90°) = -\cos x$

Recognizing the sin difference identity, you get:

$\sin x \cdot \cos 90° - \cos x \cdot \sin 90$

solve the sin + cos w/ known θ:

$\sin x \cdot 0 - \cos x \cdot 1 = -\cos x$

$0 - \cos x$

$-\cos x = -\cos x$

J is for |ˈjen(ə)rəl ˌapləˈkāSH(ə)n| (general application)

"We will shift our emphasis more toward theory and proof… often with no immediate concern for real-world relevance at all" (Franklin D. Demana et al. 404)

Before you dramatically exit the room, please read the following:

Then You Like Math!

And this math plays a part in all those things!

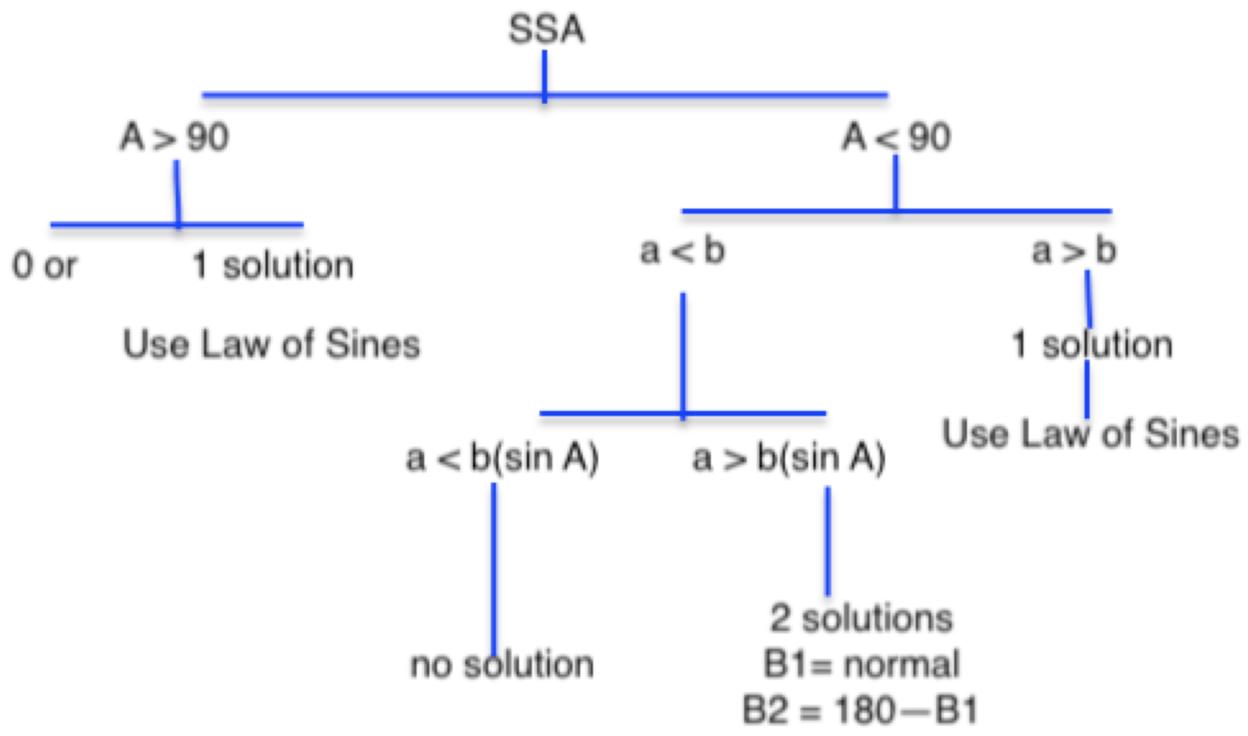
K is for the Ambiguous Case of Law of Cosine

Law of Cosines is for SAS and SSS
(Both can be used in Ambiguous case)

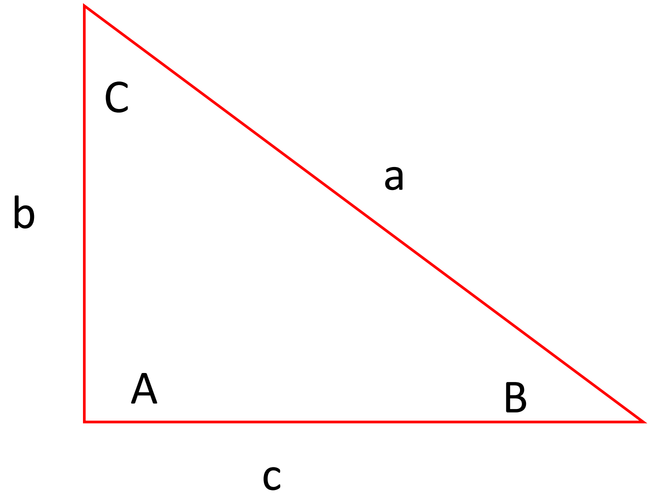
Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Similar to the Law of Sines, the Ambiguous case can be found by subtracting that angle measure from 180.



L is for Law of Sines



Sides and angles opposite each other

Check mode on calculator radians v. degrees

AAS and ASA Triangles

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Ex: A = 32 degrees, a = 17, b = 11
Find the missing parts of the triangle

$$\frac{\sin 32}{17} = \frac{\sin B}{11}$$

$$\sin^{-1}\left(\frac{11(\sin 32)}{17}\right) = 20.1°$$

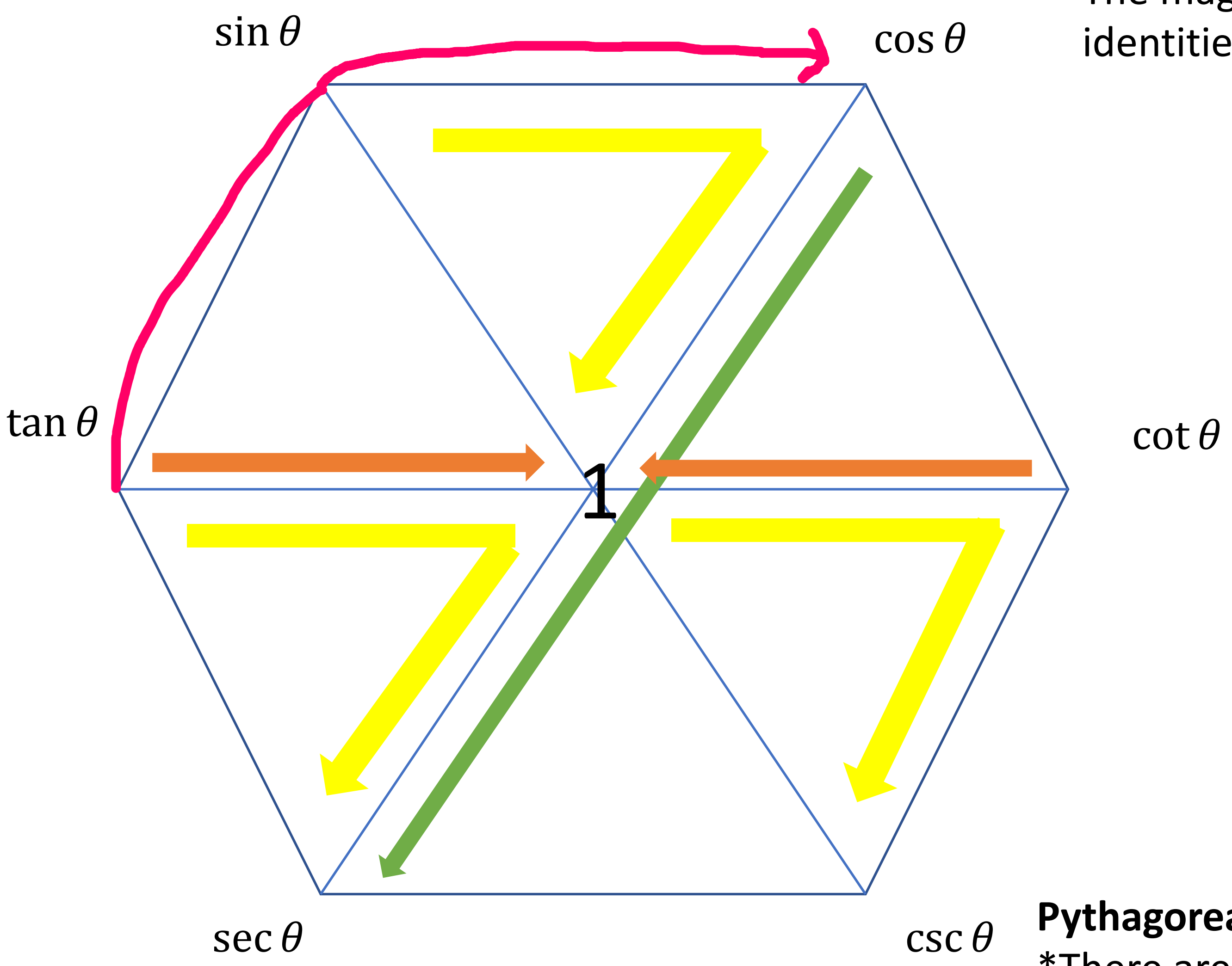
$$180° - 32° - 20° = 127.9°$$

$$\frac{\sin 32}{17} = \frac{\sin 127.9°}{c}$$

$$\frac{17(\sin 20.1)}{\sin 32} = 25.3$$

A = 32 degrees
B = 20.1 degrees
C = 127.9 degrees
a = 17
b = 11
c = 25.3

M is for Magic Hexagon



The magic hexagon can help you to visualize the relation of identities

Quotient IDs: divide clockwise and counter clockwise

$$\tan = \frac{\sin x}{\cos x}$$

Product IDs: multiply between two terms

$$\tan x \times \cot x = 1$$

Reciprocal IDs: find the reciprocal by going through 1

$$\frac{1}{\cos x} = \sec x$$

Pythagorean IDs: find by moving clockwise in triangle

$$\sin x^2 + \cos x^2 = 1$$

*other IDs can be found by dividing equation by sin or cos

$1 + cot^2x = csc^2x$

$tan^2x + 1 = sec^2x$

Pythagorean IDs = VERY IMPORTANT

*There are other IDs that can be shown on Magic Hexagon that are not shown here.

N is for Nautical Mile

Nautical mile: length of 1 minute of arc along Earth's equator

An angle measuring $\frac{1}{60}°$ intercepts an arc 1 nautical mile long and Earth's radius (at the equator) is approximately 3,956 land miles long.

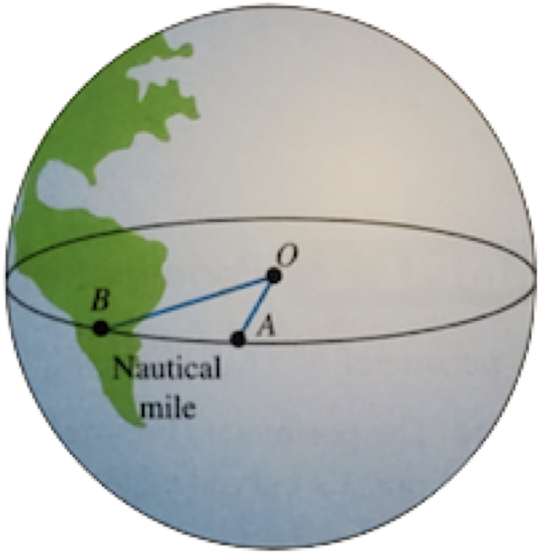
1 minute in radians

1' = $\left(\frac{1}{60}\right)° \times \frac{\pi\, rad}{180°} = \frac{\pi}{10{,}800} rad$

Using s = rθ you get the following:

1 stat mi ≈ .87 naut mi

1 naut mi ≈ 1.15 mi



Ex: The distance between two sea ports is 2,632 land miles; convert to nautical miles

$$\frac{2{,}632 \times 10{,}800}{3956\pi} \approx$$

2287.2 nautical miles

O is for the Omitted Ambiguous Case from Law of Sines (ASS)

Based on the given information, two triangles are possible

a = 6, b = 7, and A = 30 degrees

Use Law of Sines to find B
B = 35.7 degrees (arcsine will never give 2 answers for same input)

Assuming B is acute, C is 114.3 degrees and c is 10.94

If B is obtuse its measure is

$$180° - 35.7° = 144.3°$$

C and c will need to be refigured
C = $180° - 30° - 144.3° = 5.7°$

$$c = \frac{6(\sin 5.7)}{\sin 30} \approx 1.19$$

A = 30
a = 6
B = 35.7
b = 7
C = 114.3
c = 10.94

or

A = 30
a = 6
B = 144.3
b = 7
C = 5.7
c = 1.19

P is for Pythagorean Identities

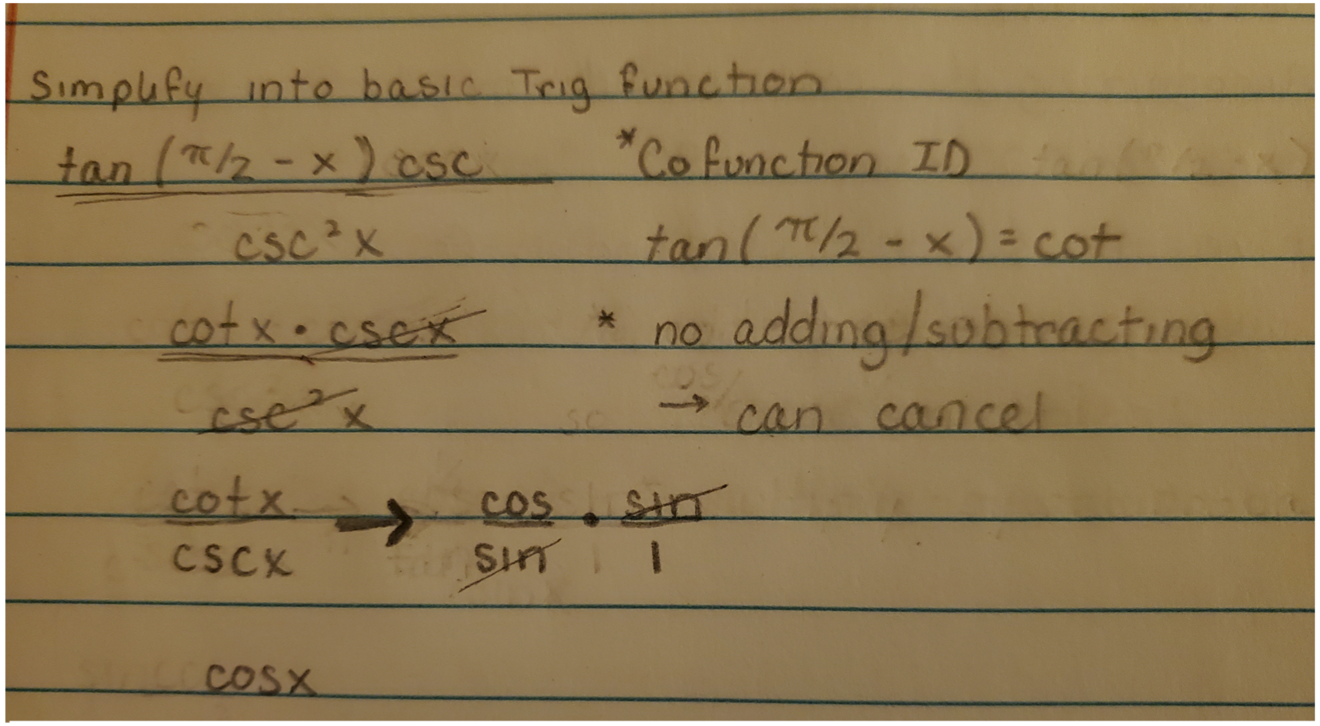
There are 3 Pythagorean Identities:

$$cos^2x + sin^2x = 1$$
$$1 + tan^2x = sec^2x$$
$$cot^2x + 1 = csc^2x$$

The second two can be formed by dividing the first

$$\frac{cos^2x + sin^2x = 1}{cos^2x} \rightarrow 1 + tan^2x = sec^2x$$

$$\frac{cos^2x + sin^2x = 1}{sin^2x} \rightarrow cot^2x + 1 = csc^2x$$



Q is for Quadrantal Angles

A quadrantal angle is an angle with its terminal side along the x or y axis
They do not form reference triangle when trying to solve the basic trig functions

$$\sin\theta = \frac{y}{r}$$

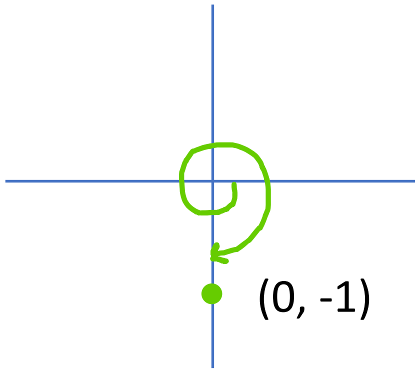
$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$

*r = radius
Radius is NEVER negative

Ex: find sin, cos, and tan -450 degrees

-450 + 360 = -90



Pick a point on the terminal side and substitute for x, y, and r

$$\sin -450° = \frac{-1}{1} = -1$$

$$\cos -450° = \frac{0}{1} = 0$$

$$\tan -450° = \frac{-1}{0} = \text{undefined}$$

R is for Converting Rectangular and Polar Equations

*polar mode in calculator

By remembering certain identities, you can change rectangular and polar equations between the two.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$

Ex: $r = 4\cos x$

By multiplying both sides by r, we are able to reduce it

$$(r = 4\cos x) \times r$$
$$r^2 = 4r\cos x$$

r^2 is the same as $x^2 + y^2$ and $r\cos x$ is the same as x

$$x^2 + y^2 = 4x$$

You can then complete the square

$$x^2 - 4x + 4 + y^2 = 4$$
$$(x-2)^2 + y^2 = 4$$

If necessary, you can rearrange the equation

Ex: $r\sec x = 3$

Remember secant is 1 divided by cosine

$$r = \frac{3}{\cos x}$$
$$r = 3\sec x$$

S is for Sinusoidal Functions

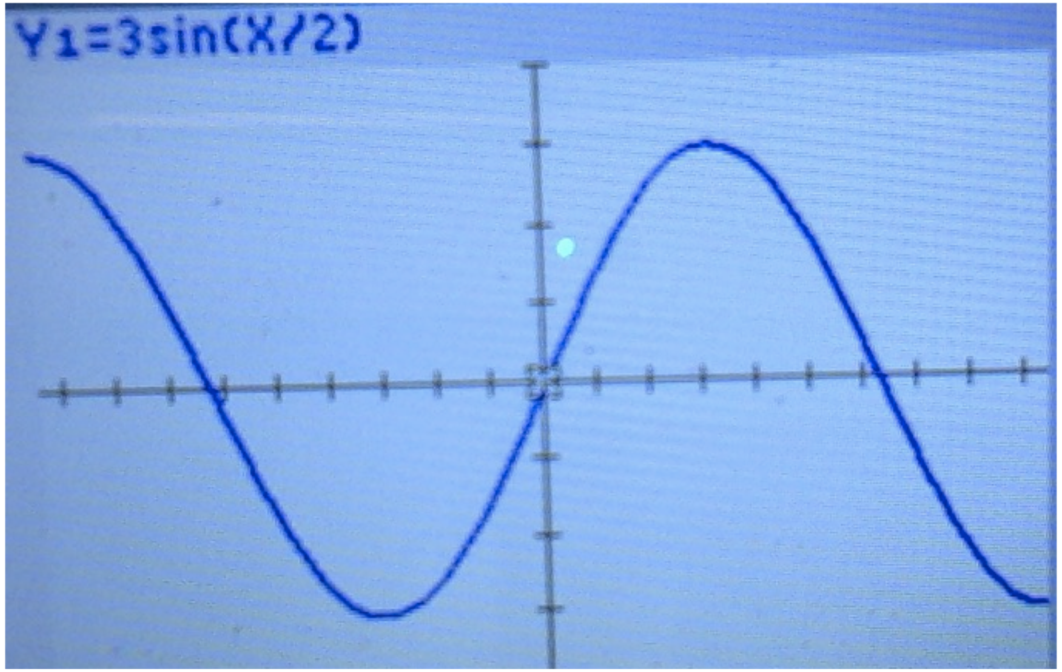
A function is a sinusoid if it can be written in the form of
$f(x) = a\sin(bx + c) + d$ and $f(x) = a\cos(bx + c) + d$

The amplitude is $|a|$
The amplitude is half the height of the wave
The period is $\frac{2\pi}{|b|}$
The period is the length of a full cycle
The frequency is $\frac{|b|}{2\pi}$
The frequency is number of complete cycles in a unit interval

Ex:

$$3\sin\frac{x}{2}$$

Amp: 3
Period: 4π
Frequency: $\frac{1}{4}\pi$



T is for Trig form to Complex form

Complex form $z = a + bi$
Trig form is polar form

$$z = r(\cos\theta, i\sin\theta)$$
$$a = r\cos\theta$$
$$b = r\sin\theta$$
$$r = \sqrt{a^2 + b^2}$$

Ex: $2 + 2i \; 0 \le \theta < 2\pi$

Find r $\sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$
To find $\theta \; \rightarrow \; \tan^{-1}\frac{b}{a}$ = 45°
Solution:

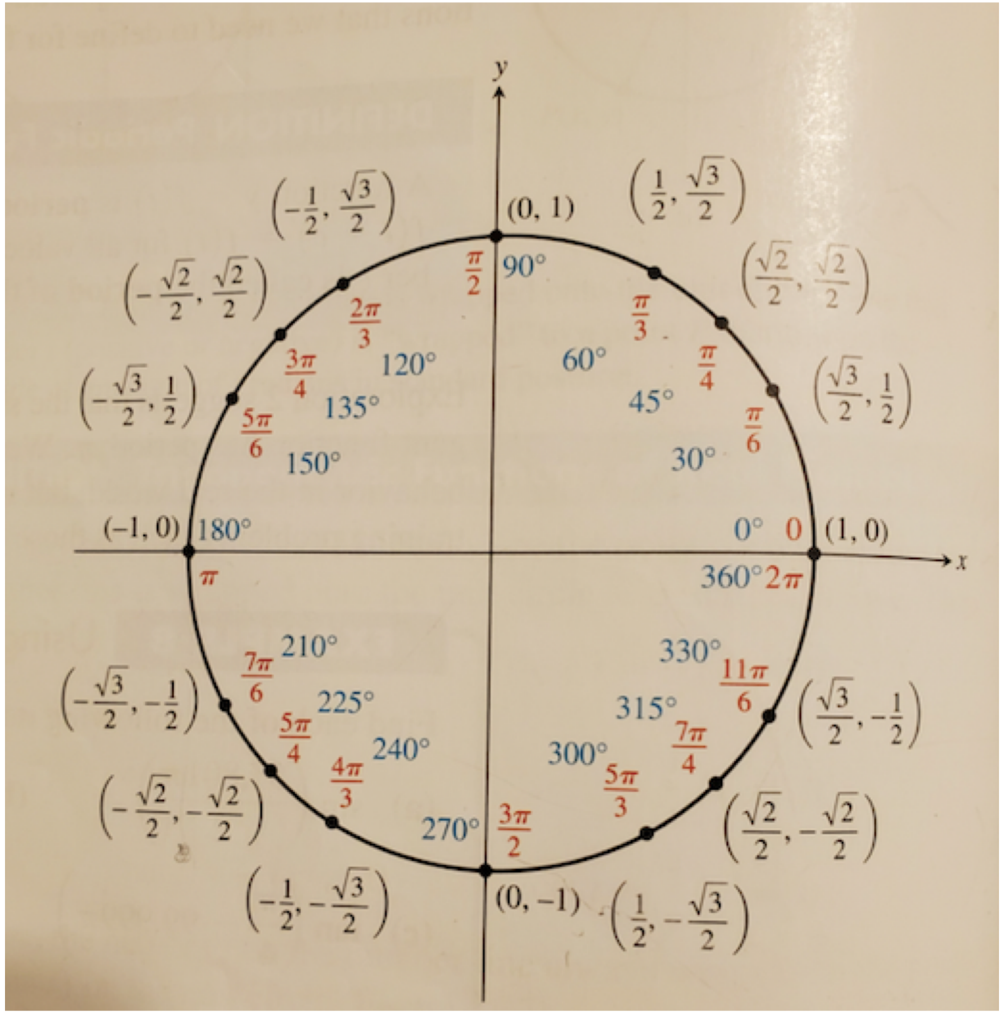
$$2\sqrt{2}(\cos 45 + i\sin 45)$$

Ex: $3(\cos 30 - i\sin 30)$

Distribute r and solve the sin and cos and you are left with

$$2.6 - i\;1.5$$

U is for Unit Circle



The Unit Circle contains conversions between degrees and radians for integer multiples of 30° & 45° and their respective coordinates

Ex: $-\frac{\pi}{3}$ $\left(\frac{5\pi}{3}\right)$ is $300°\ or\ \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$225°\ is\ \frac{5\pi}{4}\ or\ \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

These coordinates come from finding the sin and cos of these points

To know when which is +/-, remember

sin is y values

cos is x values

tan is y/x

-x, +y	+x, +y
-x, -y	+x, -y

V is for Vector

$\langle a, b\rangle$ are components of a vector
Magnitude is length of the arrow $\sqrt{a^2 + b^2}$

To add vectors add the a's and add the b's

u = $\langle -2, 4\rangle$ and v = $\langle 3, 5\rangle$

$$\langle -2 + 3, 4 + 5\rangle = \langle 1, 9\rangle$$

To subtract, do the same

u = $\langle -2, 4\rangle$ and v = $\langle 3, 5\rangle$

$$\langle -2 - 3, 4 - 5\rangle = \langle -5, -1\rangle$$

Multiplication involves SCALAR (to multiply the vector by a factor)
The same is true with division (multiplication by fraction)

2u = $\langle -4, 8\rangle$

$$\frac{1}{2}u = \langle -1, 2\rangle$$

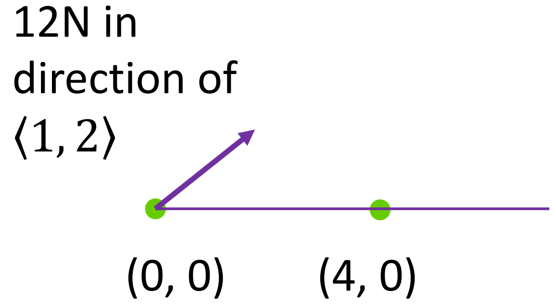
W is for Work

Work is $W = |\boldsymbol{F}| \cdot AB$
F is a constant force in a direction

F can be found by projecting F onto v

$$\frac{(F \cdot v)}{|v|^2} v$$

$|v|$ is magnitude



Ex: Find the work done by force F of 12 N acting in the direction $\langle 1, 2\rangle$ in moving an object 4m from (0, 0) to (4, 0)

$$F = 12 \frac{\langle 1, 2\rangle}{|\langle 1, 2\rangle|}$$

$$\frac{12}{\sqrt{5}}\langle 1, 2\rangle$$

Because the object is moving from (0, 0) to (4, 0), AB is $\langle 4, 0\rangle$
Now take the dot product of $\langle 1, 2\rangle\ and\ \langle 4, 0\rangle$

$12\frac{4}{\sqrt{5}}$ now take 12 times 4 and get $\frac{48}{\sqrt{5}}$

$$\approx 21.47\ lbs$$

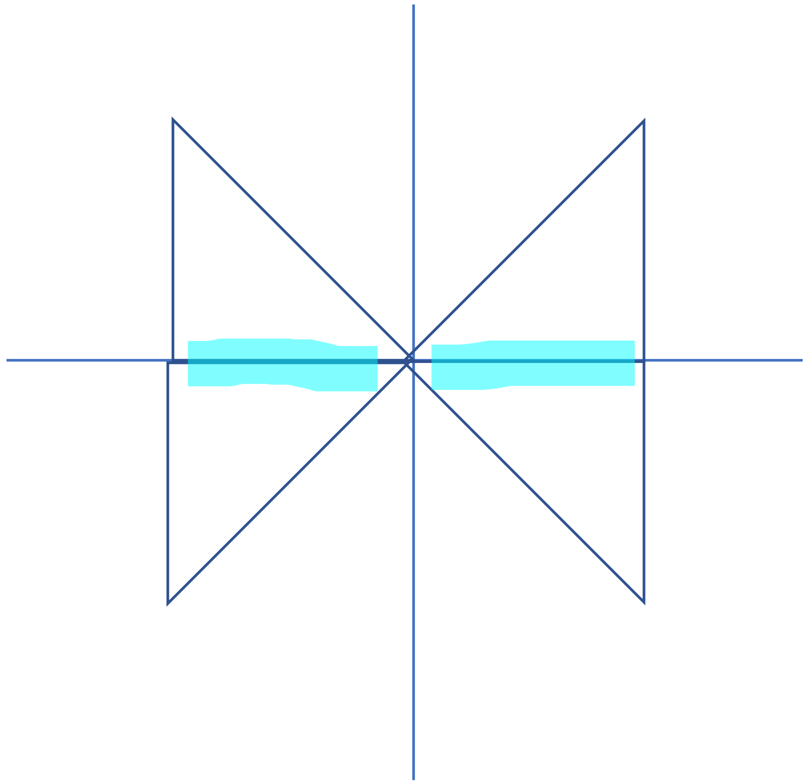
X is for Cosine

Throughout this presentation, several formulas were given and in many, cos was either given as the x variable or described as the value of a different variable.

This is because of how cos is defined. It is the x coordinate of a point over r (the distance from the origin) so when coordinates are involved, cos is the x.

In SOHCAHTOA, the hypotenuse is the length, and like r, cannot be negative Cosine is the adjacent side and therefore parallel to the x axis.

Because of its relation to x, cosine is positive in Quadrants I and IV



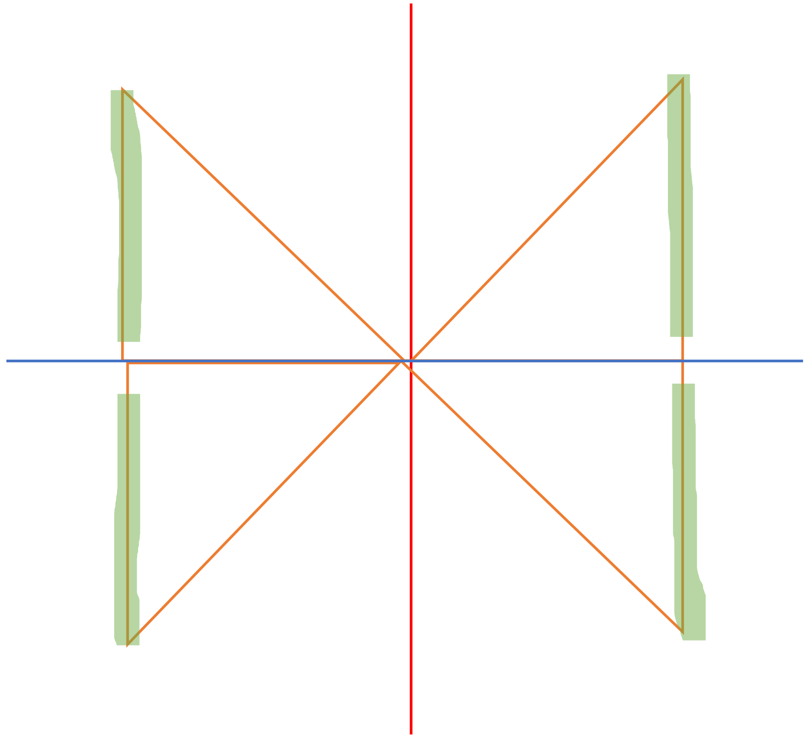
Y is for Sine

Throughout this presentation, several formulas were given and in many sin was either given as the y variable or described as the value of a different variable.

This is because sin is defined as the y coordinate of a point over r (the distance from the origin)

SOH from SOHCAHTOA is the opposite over the hypotenuse. The hypotenuse (r) is always positive and the opposite is a segment parallel to the y axis

Sin is positive in Quadrants I and II thanks to its relation to y



In every quadrant, the opposite side is a vertical line

Z is for Zero Vector

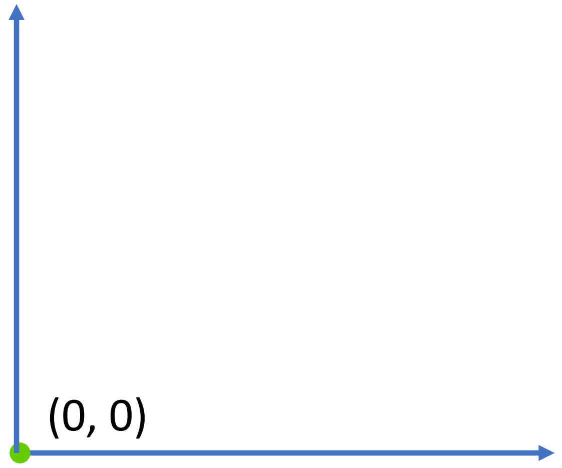
Zero vectors have no length or direction and is represented by:

$$\langle 0, 0 \rangle$$

All components are zero

The head and tail are 0

Magnitude is 0



Ex: If an apple fell from a tree and its path could be traced graphically, it would have direction and magnitude until it hit the ground